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Minimal time for the null controllability of parabolic systems: the effect of the condensation index of complex sequences

Farid Ammar Khodja Assia Benabdallah Manuel González-Burgos
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Keywords. Null controllability; parabolic systems; minimal time; condensation index of complex sequences

Abstract

Let $(\mathcal{A}, D(\mathcal{A}))$ a diagonalizable generator of a C^0-semigroup of contractions on a complex Hilbert space $\mathbb{X}$, $\mathcal{B} \in \mathcal{L}(\mathbb{C}, Y)$, Y being some suitable extrapolation space of $\mathbb{X}$, and $u \in L^2(0,T;\mathbb{C})$. Under some assumptions on the sequence of eigenvalues $\Lambda = \{\lambda_k\}_{k\geq 1} \subset \mathbb{C}$ of $(\mathcal{A}, D(\mathcal{A}))$, we prove the existence of a *minimal* time $T_0 \in [0, \infty]$ depending on Bernstein's condensation index of Λ and on $\mathcal{B}$ such that $y' = \mathcal{A}y + \mathcal{B}u$ is null-controllable at any time $T > T_0$ and not null-controllable for $T < T_0$. As a consequence, we solve controllability problems of various systems of coupled parabolic equations. In particular, new results on the boundary controllability of one-dimensional parabolic systems are derived. These seem to be difficult to achieve using other classical tools.

Contents

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1 Introduction

The starting point of this paper is to deal with the controllability properties of non-scalar parabolic systems. Before describing the problem under consideration, let us recall some known results about the controllability properties of scalar parabolic systems. The null controllability problem for scalar parabolic systems has been first considered in the one-dimensional case. Let us consider the following null controllability problem: Given $y_0 \in H^{-1}(0,\pi)$, can we find a control $v \in L^2(0,T)$ such that the corresponding solution $y \in C([0,T]; H^{-1}(0,\pi))$ to

$$\begin{cases} \partial_t y - \partial_{xx} y = 0 & \text{in } Q := (0,\pi)\times(0,T), \\ y(0,\cdot) = v, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases} \tag{1.1}$$

satisfies

$$y(\cdot,T) = 0 \text{ in } (0,\pi)? \tag{1.2}$$

Using the moment method, H. O. Fattorini and D. L. Russell gave a positive answer to the previous controllability question (see [10] and [11]). Let us briefly recall the main ideas of this moment method.

It is well-known that the operator $-\partial_{xx}$ on $(0,\pi)$ with homogenous Dirichlet boundary conditions admits a sequence of eigenvalues and normalized eigenfunctions given by

$$\mu_k = k^2, \quad \Phi_k(x) = \sqrt{\frac{2}{\pi}}\sin kx, \quad k \geq 1, \quad x \in (0,\pi), \tag{1.3}$$

and the sequence $\{\Phi_k\}_{k\geq 1}$ is a Hilbert basis of $L^2(0,\pi)$. Given $y_0 \in H^{-1}(0,\pi)$, there exists a control $v \in L^2(0,T)$ such that the solution y to (1.1) satisfies (1.2) if and only if there exists $v \in L^2(0,T)$ satisfying

$$-\langle y_0, e^{-\mu_k T}\Phi_k\rangle_{H^{-1}(0,\pi),H_0^1(0,\pi)} = \int_0^T v(t) e^{-\mu_k (T-t)} \partial_x \Phi_k(0)\, dt, \quad \forall k \geq 1.$$

Using the Fourier decomposition of y_0, $y_0 = \sum_{k\geq 1} y_{0k}\Phi_k$, this is equivalent to the existence of $v \in L^2(0,T)$ such that

$$k\sqrt{\frac{2}{\pi}}\int_0^T e^{-\mu_k(T-t)} v(t)\, dt = -e^{-\mu_k T} y_{0k}, \quad \forall k \geq 1.$$

This problem is called a *moment problem*. In [10] and [11], the authors solved the previous moment problem by proving the existence of a biorthogonal family $\{q_k\}_{k\geq 1}$ to $\{e^{-\mu_k t}\}_{k\geq 1}$ in $L^2(0,T)$ which, in particular, satisfies the additional property: for every $\epsilon > 0$ there exists a constant $C(\epsilon, T) > 0$ such that

$$\|q_k\|_{L^2(0,T)} \leq C(\epsilon, T) e^{\epsilon \mu_k}, \quad \forall k \geq 1. \tag{1.4}$$

The control is obtained as a linear combination of $\{q_k\}_{k\geq 1}$, that is,

$$v(T-s) = \sqrt{\frac{\pi}{2}} \sum_{k\geq 1} \frac{1}{k} e^{-\mu_k T} y_{0k} q_k(s)$$

and the previous bounds (1.4) are used to prove that this series converges in $L^2(0,T)$ for any positive time T. In fact, in [10] and [11] the authors proved a general result on existence of a biorthogonal family to $\{e^{-\Lambda_k t}\}_{k\geq 1}$ in $L^2(0,T)$ which fulfils appropriate bounds if the sequence $\Lambda = \{\Lambda_k\}_{k\geq 1} \subset \mathbb{R}_+$ satisfies

$$\sum_{k\geq 1} \frac{1}{\Lambda_k} < \infty \quad \text{and} \quad |\Lambda_k - \Lambda_l| \geq \rho |k - l|, \quad \forall k, l \geq 1, \tag{1.5}$$

for a constant $\rho > 0$.

It is interesting to point out how the assumptions in (1.5) are used in order to get the null controllability result for System (1.1):

1. The convergence of the previous series implies that the sequence $\{e^{-\Lambda_k t}\}_{k\geq 1}$ is not total in $L^2(0,T;\mathbb{C})$ and forms a strongly independent set in $L^2(0,T)$. In fact, this assumption assures the existence of a biorthogonal family to $\{e^{-\Lambda_k t}\}_{k\geq 1}$ in $L^2(0,T)$.

2. The previous *gap* property in (1.5) for the sequence Λ is crucial for obtaining property (1.4) and the null controllability result for System (1.1) for arbitrary small times T.

In 1973, S. Dolecki addressed the pointwise controllability at time T of the one-dimensional heat equation (see [9]). That is to say: Given $T > 0$ and $y_0 \in H^{-1}(0,\pi)$, can we find a control $v \in L^2(0,T)$ such that the solution $y \in C([0,T]; H^{-1}(0,\pi))$ of

$$\begin{cases} \partial_t y - \partial_{xx} y = \delta_{x_0} v(t) & \text{in } Q, \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases} \tag{1.6}$$

satisfies (1.2)? Here $x_0 \in (0,\pi)$ is a given point and δ_{x_0} is the Dirac distribution at this point x_0. Using again the existence of a biorthogonal family in $L^2(0,T)$ to the exponentials $\{e^{-\mu_k t}\}_{k\geq 1}$ and the bounds (1.4), S. Dolecki exhibited a minimal time T_0 such that System (1.6) is not null controllable at time T if $T < T_0$ and is null controllable at time T when $T > T_0$. This minimal time T_0 in some way "measures" the asymptotic behavior of

$$\langle \delta_{x_0}, \Phi_k \rangle_{H^{-1}(0,\pi), H_0^1(0,\pi)} = \Phi_k(x_0), \quad \forall k \geq 1,$$

with respect to the eigenvalues μ_k. Of course, this minimal time depends on the point x_0. To our knowledge, this was the first result on null controllability of parabolic problems where a minimal time of control appears. Let us emphasize that the results of [10], [11] and [9] strongly use the gap property satisfied by the eigenvalues of the Laplace operator (second property of (1.5)).

The extension of these results to systems of parabolic equations is then a natural question. In the case of boundary control, the simplest form of these systems is given by

$$\begin{cases} \partial_t y - (D\partial_{xx} + A)y = 0 & \text{in } Q, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases} \tag{1.7}$$

Here, $D = \operatorname{diag}(d_1, \ldots, d_n)$, with $d_i > 0$ for $i : 1 \leq i \leq n$, $A = (a_{ij})_{1 \leq i,j \leq n} \in \mathcal{L}(\mathbb{R}^n)$ and $B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$. In System (1.7), $v \in L^2(0,T;\mathbb{R}^m)$ is the control and we want to control the complete system (n equations) by means of m controls exerted on the boundary condition at point $x = 0$. Observe that the most interesting (and difficult) case is the case $m < n$.

The first results of null controllability for System (1.7) was obtained in [12] in the case $n = 2$, $m = 1$ and $D = Id$. This result was generalized by [3] to the case $n \geq 2$, $m \geq 1$ and $D = Id$. In these two papers, the authors used the method of moments of Fattorini-Russell to give a necessary and sufficient condition of null controllability at any time $T > 0$ for System (1.7). This condition is a generalization to non-scalar parabolic systems of the well-known Kalman rank condition for controllability of linear ordinary differential systems (see [21, Chapter 2, p. 35]). In this case the difficulty comes, firstly, from the fact that the matrix operator $\mathcal{A} := Id\partial_{xx} + A$ has eigenvalues with (geometric or algebraic) multiplicity greater than 1 and, secondly, from having less controls than equations ($m < n$). To overcome both difficulties, the authors extend the results in [10, 11], construct a biorthogonal family to $\{t^j e^{-\lambda t},\ j \in J,\ \lambda \in \Lambda\}$ in $L^2(0,T;\mathbb{C})$ (J is a finite subset of $\mathbb{N}$ and $\Lambda \subset \mathbb{C}$ is the set of complex eigenvalues of $-\mathcal{A}$) and estimate the $L^2(0,T;\mathbb{C})$-norm of its elements. In both cases, the eigenvalues of the matrix operator $\mathcal{A}$ with Dirichlet boundary conditions continue to satisfy the gap condition in (1.5). As in the scalar case (see System (1.1)), this gap property (together with appropriate properties for the coupling and control matrices A and B) provides the null controllability result for System (1.7) at any positive time.

The main motivation of this work is the extension of the previous null controllability results for System (1.7) to the case where $D \neq Id$, $n > 1$ and $m < n$. The main difference with the case $D = Id$ lies in the behavior of the sequence of eigenvalues of the matrix operator $\mathcal{A} := D\partial_{xx} + A$. Even in simple cases, the operator $-\mathcal{A}$ admits a complex sequence of eigenvalues $\Lambda = \{\Lambda_k\}_{k \geq 1}$ which does not satisfy the gap condition appearing in (1.5). Even so, following the work [3], we will see that, under appropriate assumptions (see (2.2)) and for any positive time T, it is possible to prove the existence of a biorthogonal family $\{q_k\}_{k \geq 1}$ to $\{e^{-\Lambda_k t}\}_{k \geq 1}$ in $L^2(0,T;\mathbb{C})$ but, in general, this family does not satisfy (1.4) (with $\Re(\Lambda_k)$ instead of μ_k). As a consequence, we will see that a minimal time of control $T_0 \in [0, +\infty]$ appears in such a way that System (1.7) is not null controllable at any time $T < T_0$. Let us mention the work [25] where one can find the first example of matrices D, A and B for which the minimal time is $T_0 = +\infty$ and, therefore, System (1.7) is not null controllable at any positive time T. It is interesting to notice that the system treated in [25] is approximately controllable at any positive time T.

In some situations, the boundary controllability problem (1.7) is a particular case of more abstract control problems of the following form:

$$\begin{cases} y' = \mathcal{A}y + \mathcal{B}u & \text{on } (0,T) \\ y(0) = y_0, \end{cases} \tag{1.8}$$

where:

- $\mathcal{A}$ is such that $-\mathcal{A}$ is the generator of a C^0-semigroup of contractions on a complex Hilbert space $\mathbb{X}$ whose eigenfunctions form a Riesz basis in $\mathbb{X}$ and whose sequence of eigenvalues $\Lambda = \{\lambda_k\}_{k\geq 1} \subset \mathbb{C}$ satisfies (2.2).
- $\mathcal{B} \in \mathcal{L}(\mathbb{C}, Y)$, Y being some suitable extrapolation space of $\mathbb{X}$, and $u \in L^2(0, T; \mathbb{C})$ is the control.

In this paper, we study the controllability properties of System (1.8). Let us remark that the eigenvalues of the diagonalizable operator $-\mathcal{A}$, in general, do not satisfy the gap condition in (1.5) and so, the results in [10] and [11] cannot be applied.

In our work, we obtain two kinds of controllability results for System (1.8). Firstly, we exhibit a *Kalman type condition* that ensures the approximate controllability of System (1.8) at time T. This condition is independent of T and only depends on $\mathcal{B}$ and the eigenfunctions of $\mathcal{A}$. Secondly, assuming the previous Kalman condition, we establish the existence of a minimal time $T_0 \in [0, \infty]$ (see (2.13)) such that System (1.8) is null controllable in $\mathbb{X}$ at time T if $T > T_0$ and is not null controllable in $\mathbb{X}$ at time T when $T < T_0$. Again, as in [9], we obtain a minimal time of controllability for the abstract parabolic problem (1.8). This is, in fact, the main result in this paper (Section 2).

The previous minimal time T_0 is related to the operator $\mathcal{B}$ and to the so-called *condensation index* of the sequence Λ of eigenvalues of the operator $-\mathcal{A}$, $c(\Lambda)$. To our knowledge, this condensation index has been introduced for the first time by Vl. Bernstein (see [5]) for increasing real sequences and later extended by J. R. Shackell (see [27]) to complex sequences. The goal of these two authors was to study the domain of overconvergence of Dirichlet series with real or complex exponents. Roughly speaking, if we consider the complex sequence $\Lambda = \{\lambda_k\}_{k\geq 1} \subset \mathbb{C}$, the condensation index of Λ, $c(\Lambda)$, is a measure of the way how λ_n approaches λ_m for $n \neq m$ and it is interesting to observe that if, in addition, the sequence Λ satisfies the second condition of (1.5), then $c(\Lambda) = 0$.

In some recent works, Jacob et al. (see [18] and [19]) also give a minimal time which provides the null controllability properties of abstract systems of the kind of (1.8). This authors provide a different characterization of this minimal controllability time T_0 associated with the system. The construction of this time T_0 seems to be less explicit than ours. The method of proof of the corresponding positive and negative controllability results is also different: their arguments turn around Carleson measures while ours make use of the condensation index associated with the sequence of eigenvalues of the operator associated with the abstract parabolic problem.

The proof of our main result is divided into two parts. Firstly, we prove the positive null controllability result for System (1.8) (see Section 5.1) using the moment method and following some ideas from [10], [11] and [3] (see Section 4). The second part is devoted to proving the negative null controllability result (Section 5.2). To this end, we carry out a refined study of the condensation index associated to a class of complex sequences (see Section 3).

In this paper we also give some applications of our main result to scalar and non-scalar parabolic problems with distributed and boundary controls (Section 6). First, we generalize the controllability result for the scalar heat equation stated in [9]. Secondly, we provide a complete null controllability result for System (1.7) in the simplest (but non-trivial) case $n = 2$, $m = 1$,

$$D = \mathrm{diag}\,(1, d),\ d > 0, \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

In this case, the minimal time, T_0, only depends on $c(\Lambda)$, the condensation index of the sequence Λ of eigenvalues of the matrix operator $-D\Delta - A$ associated with homogeneous Dirichlet boundary conditions, i.e., $\Lambda = \{k^2, dk^2\}_{k\geq 1}$. In fact, we will prove that T_0 strongly depends on the diophantine approximation properties of the irrational number d (see Section 6.2). As third application, we will consider the null controllability problem for system

$$\begin{cases} \partial_t y - (D\partial_{xx} + A)\, y = f(x)Bv(t) & \text{in } Q, \\ y(0,\cdot) = y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

where D, A and B are as before, $f \in H^{-1}(0,\pi)$ is a given function and $y_0 \in L^2(0,\pi;\mathbb{R}^2)$ is the initial datum. Observe that $v \in L^2(0,T)$ is a scalar control and we want to control the 2×2 system with one control force. In this case we will see that the minimal time of control T_0 depends on the same previous condensation index and on the function f.

Up to now we have described some controllability problems (with distributed or boundary controls) for scalar or non-scalar parabolic systems in the one-dimensional case. Let us briefly provide a non-extensive literature on the corresponding results in the N-dimensional case ($N \geq 2$).

The N-dimensional null controllability problem for scalar parabolic equations (with boundary or distributed controls) was independently solved by G. Lebeau and L. Robbiano, [24] (for the heat equation), and A. Fursikov and O. Imanuvilov, [13] (for a general parabolic equation). The result in [24] was obtained through a spectral inequality and this inequality was proved by the authors proving local Carleman estimates. The result in [13] was obtained by proving Carleman estimates that imply an observability inequality equivalent to the null controllability or controllability to trajectories of the parabolic equation. Carleman inequalities have been introduced by [7] for proving uniqueness results for some PDE's and have been widely extended by Hörmander (see [16, 17]). See also [23] where different Carleman inequalities are presented and compared and where some applications to the controllability of the heat equation is also done.

There are very few results in the literature concerning the boundary null controllability of coupled parabolic systems in the N-dimensional case ($N \geq 2$). In [1] and [2], the authors deal with this problem in the case of some 2×2 parabolic systems and give some sufficient conditions imposing appropriate geometric conditions. These conditions are inherited from the method, that consists in proving a result for coupled hyperbolic equations and then, using the Kannai transform, they obtain the result for parabolic equations. For a survey of recent results on null controllability (with boundary or distributed controls) in the framework of non-scalar parabolic systems, see [4] and the references therein.

In this paper we have treated the null controllability problem for System (1.8) when the operator $\mathcal{A}$ is, among other thinks, diagonalizable, i.e., its eigenfunctions form a Riesz basis in $\mathbb{X}$. In a forthcoming paper we will address the controllability problem for this system when $\mathcal{A}$ admits a Riesz basis in $\mathbb{X}$ of eigenfunctions and generalized eigenfunctions. This will be crucial in order to study the controllability properties of System (1.7) in the general case.

The plan of the paper is the following one: In Section 2, we address some preliminary results and we give the main result of this work. In Section 3 we study the so-called condensation index of complex sequences $\Lambda = \{\lambda_k\}_{k\geq 1}$ which satisfy (2.2). This section is crucial in order to prove the negative controllability result stated in the main result. Section 4 is devoted to the construction and estimates of a biorthogonal family to complex exponentials. We will use the results of this section for proving the positive null controllability part of the main result. In Section 5 we give the proof of the main result. In Section 6 we exhibit some applications of the main result to the null controllability problem for scalar and non-scalar parabolic systems. Finally, in the Appendix we prove some technical results from the diophantine approximation theory of irrational numbers.

2 Preliminaries and main result

Let $\mathbb{X}$ be a Hilbert space on $\mathbb{C}$ with norm and inner product respectively denoted by $\|\cdot\|$ and $(\cdot,\cdot)$. Let us also consider $\{\phi_k\}_{k\geq 1}$ a Riesz basis of $\mathbb{X}$ and let us denote $\{\psi_k\}_{k\geq 1}$ the corresponding biorthogonal sequence to $\{\phi_k\}_{k\geq 1}$. Let us recall that if $y \in \mathbb{X}$, then

$$y = \sum_{k\geq 1} (y, \psi_k)\, \phi_k \quad \text{and} \quad \|y\|_0^2 := \sum_{k\geq 1} |(y,\psi_k)|^2 < \infty, \tag{2.1}$$

and

$$y = \sum_{k\geq 1} (y, \phi_k)\, \psi_k \quad \text{and} \quad (\|y\|_0^*)^2 := \sum_{k\geq 1} |(y,\phi_k)|^2 < \infty.$$

In fact, $\|\cdot\|_0$ and $\|\cdot\|_0^*$ define in $\mathbb{X}$ equivalent norms to the usual norm $\|\cdot\|$.

Let us also consider a sequence $\Lambda = \{\lambda_k\}_{k\geq 1} \subset \mathbb{C}$ satisfying

$$\begin{cases} \lambda_i \neq \lambda_k, \quad \forall i,k \in \mathbb{N} \text{ with } i \neq k, \\ \Re(\lambda_k) \geq \delta\, |\lambda_k| > 0, \quad \forall k \geq 1, \quad \text{and} \quad \displaystyle\sum_{k\geq 1} \frac{1}{|\lambda_k|} < \infty, \end{cases} \tag{2.2}$$

for a positive constant δ.

We denote by $\mathbb{X}_{-1}$ (resp., $\mathbb{X}_{-1}^*$) the completion of $\mathbb{X}$ with respect to the norm:

$$\|y\|_{-1} := \left(\sum_{k\geq 1} \frac{|(y,\psi_k)|^2}{|\lambda_k|^2} \right)^{1/2}, \quad \forall y \in \mathbb{X},$$

(resp.,

$$\|y\|_{-1}^* := \left(\sum_{k\geq 1} \frac{|(y,\phi_k)|^2}{|\lambda_k|^2} \right)^{1/2}, \quad \forall y \in \mathbb{X}).$$

On the other hand, the Hilbert space $(\mathbb{X}_1, \|\cdot\|_1)$ (resp. $(\mathbb{X}_1^*, \|\cdot\|_1)$) is defined by

$$\mathbb{X}_1 := \{ y \in \mathbb{X} : \|y\|_1 < \infty \} \text{ with } \|y\|_1 = \left(\sum_{k\geq 1} |\lambda_k|^2 \, |(y,\psi_k)|^2 \right)^{1/2},$$

(resp.,

$$\mathbb{X}_1^* := \{ y \in \mathbb{X} : \|y\|_1^* < \infty \} \text{ with } \|y\|_1^* = \left(\sum_{k\geq 1} |\lambda_k|^2 \, |(y,\phi_k)|^2 \right)^{1/2}).$$

It is well-known (see for instance [29]) that $\mathbb{X}_{-1} = (\mathbb{X}_1^*)'$, the dual space of $\mathbb{X}_1^*$ with respect to the pivot space $\mathbb{X}$.

Let us fix $T > 0$ a real number. We consider a system of the form:

$$\begin{cases} y' = \mathcal{A}y + \mathcal{B}u \quad \text{on } (0,T) \\ y(0) = y_0 \in \mathbb{X}. \end{cases} \tag{2.3}$$

In the previous system we will assume that $\mathcal{A} : D(\mathcal{A}) \subset \mathbb{X} \to \mathbb{X}$ is the operator given by:

$$\mathcal{A} = -\sum_{k\geq 1} \lambda_k \left(\cdot, \psi_k\right) \phi_k, \tag{2.4}$$

with $D(\mathcal{A}) = \mathbb{X}_1$. Under assumptions (2.2), we can readily prove that $\mathcal{A}$ is densely defined and is invertible with $\mathcal{A}^{-1} \in \mathcal{L}(\mathbb{X})$. It is also easy to check that:

$$\mathcal{A}^* = -\sum_{k\geq 1} \overline{\lambda}_k \left(\cdot, \phi_k\right) \psi_k, \tag{2.5}$$

with $D(\mathcal{A}^*) = \mathbb{X}_1^*$.

The operator $\mathcal{A}$ admits an extension $\mathcal{A}_{-1} \in \mathcal{L}\left(\mathbb{X}, \mathbb{X}_{-1}\right)$ and we will denote it by the same symbol $\mathcal{A}$. The C^0-semigroup $\left\{e^{t\mathcal{A}}\right\}_{t>0}$ generated by $\mathcal{A}$ on $\mathbb{X}$ also extends to a C^0-semigroup on $\mathbb{X}_{-1}$ generated by (the extension of) $\mathcal{A}$. We will still denote this semigroup by $\left\{e^{t\mathcal{A}}\right\}_{t>0}$.

In System (2.3), $u \in L^2\left(0, T; \mathbb{C}\right)$ is the control which acts on the system by means of the operator $\mathcal{B} \in \mathcal{L}\left(\mathbb{C}, \mathbb{X}_{-1}\right)$ (thus $\mathcal{B}^* \in \mathcal{L}\left(\mathbb{X}_1^*, \mathbb{C}\right) \equiv \mathbb{X}_{-1}$). In the sequel, we will suppose that $\mathcal{B}$ is an admissible control operator for the semigroup generated by $\mathcal{A}$, i.e., for a positive time T^* one has

$$\mathrm{R}\left(L_{T^*}\right) \subset \mathbb{X},$$

where

$$L_T u = \int_0^T e^{(T-s)\mathcal{A}} \mathcal{B} u(s)\, ds, \quad \forall u \in L^2\left(0, T; \mathbb{C}\right). \tag{2.6}$$

Observe that as a consequence of the closed graph theorem, the previous assumption implies that $L_T \in \mathcal{L}(L^2(0,T;\mathbb{C}), \mathbb{X})$. Moreover, it can be checked that $L_T^* \in \mathcal{L}(\mathbb{X}, L^2(0,T;\mathbb{C}))$ is given by

$$L_T^* \varphi_0 = \mathcal{B}^* e^{(T-t)\mathcal{A}^*} \varphi_0, \quad \forall \varphi_0 \in \mathbb{X}.$$

The mild solution to (2.3) is given by

$$y(t) = e^{t\mathcal{A}} y_0 + \int_0^t e^{(t-s)\mathcal{A}} \mathcal{B} u(s)\, ds, \quad t \in (0, T) \tag{2.7}$$

and under the admissibility assumption on the operator $\mathcal{B}$, one has $y \in C^0([0,T]; \mathbb{X})$.

We recall that:

Definition 2.1. It will be said that System (2.3) is approximately controllable in $\mathbb{X}$ at time $T > 0$ if for any $y_0, y_d \in \mathbb{X}$ and any $\varepsilon > 0$ there exists a control $u \in L^2(0,T;\mathbb{C})$ such that the solution $y \in C^0([0,T]; \mathbb{X})$ to (2.3) satisfies

$$\|y(T) - y_d\| \leq \varepsilon.$$

On the other hand, it will be said that System (2.3) is *null controllable* in $\mathbb{X}$ at time $T > 0$ if for all $y_0 \in \mathbb{X}$, there exists $u \in L^2(0,T;\mathbb{C})$ such that the solution $y \in C^0([0,T]; \mathbb{X})$ to (2.3) satisfies

$$y(T) = e^{T\mathcal{A}} y_0 + L_T u = 0.$$

Remark 2.2. It is possible to define another controllability property of System (2.3): the exact controllability to trajectories in $\mathbb{X}$ at time $T > 0$. It will be said that System (2.3) is exactly controllable to trajectories in $\mathbb{X}$ at time $T > 0$ if for any $y_0 \in \mathbb{X}$ and any trajectory $\widehat{y} \in C^0([0,T]; \mathbb{X})$, i.e., a mild solution to (2.3) associated with $\widehat{y}_0 \in \mathbb{X}$ and $\widehat{u} \in L^2(0,T;\mathbb{C})$, there exists a control $u \in L^2(0,T;\mathbb{C})$ such that the solution $y \in C^0([0,T];\mathbb{X})$ to (2.3) satisfies

$$y(T) = \widehat{y}(T) \text{ in } \mathbb{X}).$$

At first sight, this concept could seem stronger than the null controllability property but, thanks to the linear character of System (2.3), it is easy to check that the null controllability in $\mathbb{X}$ at time $T > 0$ of this system is equivalent to the exact controllability to trajectories in $\mathbb{X}$ at time T. □

It is well-known that the controllability properties of System (2.3) amount to appropriate properties of the so-called *adjoint system* to System (2.3). This adjoint system has the form:

$$\begin{cases} -\varphi' = \mathcal{A}^*\varphi, & \text{on } (0,T) \\ \varphi(T) = \varphi_0 \in \mathbb{X}. \end{cases} \tag{2.8}$$

Observe that, under the previous assumptions on the operator $\mathcal{A}$, for any $\varphi_0 \in \mathbb{X}$ System (2.8) admits a unique weak solution $\varphi \in C^0([0,T];\mathbb{X})$. One has (see for instance [29, Theorem 11.2.1] and [30, Theorems 2.5 and 2.6, p. 213]):

Theorem 2.3. *Assume that $\mathcal{B} \in \mathcal{L}(\mathbb{C}, \mathbb{X}_{-1})$ is an admissible control operator for the semigroup $\left\{e^{t\mathcal{A}}\right\}_{t>0}$ generated by $\mathcal{A}$, with $\mathcal{A}$ given by (2.4), and $\Lambda = \{\lambda_k\}_{k\ge1}$ is a complex sequence satisfying (2.2). Then,*

1. *System (2.3) is approximately controllable in $\mathbb{X}$ at time T if and only if the solutions $\varphi \in C^0([0,T];\mathbb{X})$ to the adjoint system (2.8) satisfy the unique continuation property*

   "If $\mathcal{B}^*\varphi(t) = 0$ for almost any $t \in [0,T]$, then $\varphi \equiv 0$."

2. *System (2.3) is null controllable in $\mathbb{X}$ at time T if and only if there exists a constant $C_T > 0$ such that any solution $\varphi \in C^0([0,T];\mathbb{X})$ to the adjoint system (2.8) satisfies the observability inequality*

$$\|\varphi(0)\|^2 \le C_T \int_0^T |\mathcal{B}^*\varphi(t)|^2 \, dt. \qquad \square$$

Let us now take $\varphi_0 \in \mathbb{X}$. Then, it is not difficult to check that the corresponding solution to the adjoint problem (2.8) is given by

$$\varphi(t) = \sum_{k\ge1} e^{-\overline{\lambda}_k(T-t)} \left(\varphi_0, \phi_k\right) \psi_k, \quad \forall t \in [0,T]. \tag{2.9}$$

Thus, as an easy consequence of the previous result we have:

Corollary 2.4. *Under the assumptions of Theorem 2.3, one has:*

1. *System (2.3) is approximately controllable in $\mathbb{X}$ at time T if and only if*

$$b_k := \mathcal{B}^*\psi_k \neq 0, \quad \forall k \ge 1. \tag{2.10}$$

2. *System (2.3) is null controllable in $\mathbb{X}$ at time T if and only if there exists a constant $C_T > 0$ such that*

$$\sum_{k\ge1} e^{-2T\Re(\lambda_k)} |a_k|^2 \le C_T \int_0^T \left| \sum_{k\ge1} \overline{b}_k e^{-\lambda_k(T-t)} a_k \right|^2, \quad \forall \{a_k\}_{k\ge1} \in \ell^2(\mathbb{C}). \tag{2.11}$$

Proof. 1. Taking $\varphi_0 = \psi_k$ in System (2.8), we deduce that condition (2.10) is a necessary condition for the unique continuation property for System (2.8). On the other hand, let us introduce the notation

$$e_k(t) = e^{-\lambda_k t}, \quad \forall t \in [0,T].$$

Then, condition (2.2) in particular implies that the sequence $\{e_k\}_{k\geq 1}$ is not total in $L^2(0,T;\mathbb{C})$ and forms a strongly independent set, i.e.,

$$e_n \notin \overline{\text{span}\,\{e_k : k \neq n\}}^{L^2(0,T;\mathbb{C})}, \quad \forall n \geq 1, \tag{2.12}$$

(see Remark 4.4). Let us take $\varphi_0 \in \mathbb{X}$ such that the solution φ to (2.8) satisfies $\mathcal{B}^*\varphi(\cdot) = 0$ on $(0,T)$, i.e.,

$$\mathcal{B}^*\varphi(t) = \sum_{k\geq 1} e^{-\overline{\lambda}_k(T-t)}\,(\varphi_0, \phi_k)\,\mathcal{B}^*\psi_k = 0, \quad \text{a.e. on } (0,T).$$

Property (2.12) implies $\varphi \equiv 0$ and thus the sufficient condition.

2. This point can be easily deduced from Theorem 2.3 using that $\{\psi_k\}_{k\geq 1}$ is also a Riesz basis of $\mathbb{X}$ and the expression (2.9). □

Remark 2.5. Corollary 2.4 provides two conditions which give the controllability properties of System (2.3). It is clear that the observability inequality (2.11) implies condition (2.10). Therefore, we deduce that (2.10) is a necessary condition for the null controllability property of System (2.3) at time T or, equivalently, if System (2.3) is null controllable in $\mathbb{X}$ at time $T > 0$, then this system is approximately controllable in $\mathbb{X}$ at *any positive time*. □

Corollary 2.4 provides two necessary and sufficient conditions for the approximate and null controllability properties of System (2.3). Observe that condition (2.10) only depends on $\mathcal{B}$ and the basis $\{\psi_k\}_{k\geq 1}$. This condition is independent of the final observation time T. However, we will see that the observability inequality (2.11) strongly depends on the coefficients $\{b_k\}_{k\geq 1}$ and the qualitative behavior of the sequence Λ of eigenvalues of $-\mathcal{A}$. In fact, we will prove that the null controllability result for System (2.3) also depends on the final time of observation $T > 0$ by means of some minimal controllability time T_0 that, in some sense, measures the qualitative behavior of the sequences Λ and $\{b_k\}_{k\geq 1}$.

This is our main result. It reads as follows:

Theorem 2.6. *Assume that $\mathcal{B} \in \mathcal{L}(\mathbb{C}, \mathbb{X}_{-1})$ is an admissible control operator for the semigroup $\{e^{t\mathcal{A}}\}_{t>0}$ generated by $\mathcal{A}$, with $\mathcal{A}$ given by (2.4), and $\Lambda = \{\lambda_k\}_{k\geq 1}$ is a complex sequence satisfying (2.2). Let us suppose furthermore that condition (2.10) holds. Let us introduce*

$$T_0 = \limsup \left(\frac{\log \frac{1}{|b_k|}}{\Re(\lambda_k)} + \frac{\log \frac{1}{|E'(\lambda_k)|}}{\Re(\lambda_k)} \right), \tag{2.13}$$

where

$$E(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{\lambda_k^2}\right), \quad z \in \mathbb{C}. \tag{2.14}$$

Then:

1. *System (2.3) is null controllable for $T > T_0$;*

2. *System (2.3) is not null controllable for $T < T_0$.* □

Remark 2.7. Let us give some remarks about the statement of Theorem 2.6:

1. The proof of Corollary 2.4 also shows that condition (2.10) is a necessary condition for the null controllability of System (2.3) at any time $T > 0$.

2. Thanks to assumption (2.2), we deduce that the infinite product (2.14) is uniformly and absolutely convergent on the compact sets of $\mathbb{C}$. In particular, this implies that the value of this infinite product is independent of the order of the factors. Thus, we deduce that the function E is holomorphic in $\mathbb{C}$ and independent of rearrangements of the sequence Λ.

3. Using again condition (2.2) we also deduce the property $E'(\lambda_k) \neq 0$ for all $k \geq 1$. This guarantees that T_0 given by (2.13) is well-defined.

4. We will prove that $T_0 \in [0, \infty]$ (see Theorem 4.8) and, in fact, we will see that T_0 could take any value on the interval $[0, \infty]$. Thus, when $T_0 \in (0, \infty]$, from Corollary 2.4 and Theorem 2.6 we deduce that System (2.3) could have the approximate controllability property at a positive time T wihout being null controllable at this time T. □

Theorem 2.6 establishes the existence of a *minimal time* T_0 which provides the null controllability properties for System (2.3). In the definition of this optimal time T_0 two elements intervene. The first one

$$T_1 = \limsup \frac{\log \frac{1}{|b_k|}}{\Re(\lambda_k)}$$

only depends on the sequence Λ and on the control operator $\mathcal{B}$. The second one is the *condensation index* of the sequence Λ, $c(\Lambda)$, (see Definition 3.1) and, of course, only depends on Λ. We will also see that if $T_1 = 0$ (resp., $c(\Lambda) = 0$) then $T_0 = c(\Lambda)$ (resp., $T_0 = T_1$) (see Theorem 4.8). In this sense, we will also see that Theorem 2.6 generalizes the results on null controllability proved in [10], [11] (where $T_0 = 0$, see Remark 6.27), [9] (where $T_0 = T_1$, see Subsection 6.1) and [25] (where $T_0 = c(\Lambda) = \infty$, see Subsection 6.2).

Theorem 2.6 will be proved in Subsections 5.1 and 5.2.

As we will see in the next sections, when the control operator $\mathcal{B}$ is "good" (in some sense), the minimal time T_0 coincides with $c(\Lambda)$, the condensation index of the sequence Λ. This condensation index is a measure of the separation of the elements of the complex sequence Λ. We will see that $c(\Lambda) \in [0, \infty]$ (Remark 3.10) and, if the sequence Λ satisfies the second condition in (1.5), then $c(\Lambda) = 0$ (Proposition 3.11).

Let us now present an interesting application of Theorem 2.6 to the controllability problem for System (1.7) in the simplest case $n = 2$,

$$D = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \text{ (with } d > 0\text{)}, \quad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

i.e.,

$$\begin{cases} \partial_t y_1 - \partial_{xx} y_1 = y_2 & \text{in } Q, \\ y_1(0, \cdot) = y_1(\pi, \cdot) = 0 & \text{on } (0, T), \\ y_1(\cdot, 0) = y_{0,1} & \text{in } (0, \pi), \end{cases} \qquad \begin{cases} \partial_t y_2 - d\partial_{xx} y_2 = 0 & \text{in } Q, \\ y_2(0, \cdot) = v, \quad y_2(\pi, \cdot) = 0 & \text{on } (0, T), \\ y_2(\cdot, 0) = y_{0,2} & \text{in } (0, \pi), \end{cases} \tag{2.15}$$

where Q is the cylinder $Q = (0, \pi) \times (0, T)$.

For any $v \in L^2(0,T)$ and $y_0 = (y_{0,1}, y_{0,2}) \in H^{-1}(0,\pi;\mathbb{R}^2)$ System (1.7) has a unique solution

$$y = (y_1, y_2) \in C^0([0,T]; H^{-1}(0,\pi;\mathbb{R}^2))$$

which depends continuously on the data.

As said before, the controllability properties of System (1.7) with $n = 2$, $m = 1$ and $D = Id$ are well-known (see [12]). The case $D \neq Id$ is widely open and only few results are known (see [25]) . We will see that, for System (2.15) with $d \neq 1$, it is possible to apply Corollary 2.4.1 and Theorem 2.6, with $\mathbb{X} = H^{-1}(0,\pi;\mathbb{R}^2)$, obtaining the following result:

Theorem 2.8. *Assume that $d \neq 1$. Then, one has*

1. *System (2.15) is approximately controllable in $\mathbb{X} = H^{-1}(0,\pi;\mathbb{R}^2)$ at time $T > 0$ if and only if $\sqrt{d} \notin \mathbb{Q}$.*
2. *Assume that $\sqrt{d} \notin \mathbb{Q}$. Then, there exists a time $T_d \in [0,\infty]$, which only depends on d, such that*
   (a) *System (2.15) is null controllable in $\mathbb{X} = H^{-1}(0,\pi;\mathbb{R}^2)$ at any time $T > T_d$;*
   (b) *System (2.15) is not null controllable in $\mathbb{X}$ for $T < T_d$.* □

From this result we deduce that the controllability properties of System (2.15) depend on the diffusion coefficient d via the condition $\sqrt{d} \notin \mathbb{Q}$ and the minimal time T_d. We will also see that this minimal null controllability time is in fact the condensation index associated to the sequence of eigenvalues of the operator $-(D\partial_{xx} + A)$, does not depend on B and can be explicitly computed.

In Subsection 6.2 we will also analyze the dependence of the optimal time T_d with respect to d and we will see that this dependence is, in fact, very intricate and connected with the diophantine approximation of the irrational $\sqrt{d}$:

Theorem 2.9. *Assume that $d \neq 1$. Then, one has*

1. *For any $\tau_0 \in [0,\infty]$, there exists $d \in (0,\infty)$, satisfying $\sqrt{d} \notin \mathbb{Q}$, such that $T_d = \tau_0$.*
2. *There exists $\mathcal{M} \subset (0,\infty)$, with $|\mathcal{M}| = 0$, such that $T_d = 0$ for all $d \in (0,\infty) \setminus \mathcal{M}$.*
3. *Given $\tau_0 \in [0,\infty]$, the set $\{d \in (0,\infty) : T_d = \tau_0\}$ is dense in $(0,\infty)$.*

In the previous points T_d is the minimal null controllability time associated with System (2.15) provided by Theorem 2.8. □

Theorems 2.8 and 2.9 will be proved as a consequence of the results in Subsection 6.2.

3 The condensation index of complex sequences

In this section we are going to study the so-called condensation index of a complex sequence. To this end and throughout this section, we will consider a complex sequence $\Lambda = \{\lambda_k\}_{k\geq 1} \subset \mathbb{C}$ satisfying condition (2.2). In addition, we are going to assume that the sequence Λ is normally ordered, i.e.,

$$\begin{cases} |\lambda_k| \leq |\lambda_{k+1}|, \quad \forall k \geq 1, \\ -\dfrac{\pi}{2} < \arg(\lambda_k) < \arg(\lambda_{k+1}) < \dfrac{\pi}{2} \text{ when } |\lambda_k| = |\lambda_{k+1}|. \end{cases} \tag{3.1}$$

Under the previous assumptions, let us introduce the following definition:

Definition 3.1. The *index of condensation* of a sequence $\Lambda = \{\lambda_k\}_{k\geq 1} \subset \mathbb{C}$ satisfying (2.2) is the real number

$$c(\Lambda) = \limsup \frac{\log \frac{1}{|E'(\lambda_k)|}}{\Re(\lambda_k)}. \tag{3.2}$$

where the interpolating function E is given by (2.14). □

Remark 3.2. Taking into account Remark 2.7 we deduce that the condensation index $c(\Lambda)$ is well-defined and its value is independent of rearrangements of the sequence Λ. In particular, we can always assume that the sequence Λ is normally ordered and satisfies (3.1). □

The previous definition was introduced by Vl. Bernstein in [5] for real sequences Λ. The index of condensation $c(\Lambda)$ provides a measure of the separation of the elements λ_n of the sequence Λ. This concept is strongly related to the overconvergence and location of singular points of Dirichlet series of real or complex exponents (see [5] and [27]).

Remark 3.3. Under the previous assumptions on the sequence Λ, we will see that the corresponding condensation index $c(\Lambda)$ is a non-negative number, i.e., $c(\Lambda) \in [0, \infty]$. In fact, we will provide some examples of sequences Λ for which $c(\Lambda)$ is 0, or is a finite positive number or $c(\Lambda) = \infty$. Moreover, we will give some general conditions on the sequence Λ which ensure that $c(\Lambda) = 0$, a very interesting case for the controllability properties of the parabolic system (2.3). □

Our next goal will be to get a different formula for the condensation index $c(\Lambda)$ that will be used in the proof of our main result, Theorem 2.6. To this end, we will adapt to our setting the definition of condensation index of complex sequences given by Shackell in [27]. It starts with the notion of condensation grouping associated with the sequence Λ. This concept generalizes the one introduced by Vl. Bernstein [5] for real sequences.

Definition 3.4. A sequence of sets $\Delta = \{G_k\}_{k\geq 1}$ is a *condensation grouping* of the sequence $\Lambda = \{\lambda_k\}_{k\geq 1}$ if it satisfies the following conditions:

(i) For each $k \geq 1$, the cardinal of $G_k \cap \Lambda$ is finite and equal to $p_k + 1$, for some integer $p_k \geq 0$;

(ii) If $\lambda \in \Lambda$, then there exists $k \geq 1$ such that $\lambda \in G_k$, i.e.,

$$\bigcup_{k\geq 1} G_k \cap \Lambda = \Lambda;$$

(iii) Let λ_{l_k} be the first element of $G_k \cap \Lambda$. Then the sequence $\{\lambda_{l_k}\}_{k\geq 1}$ is normally ordered and $p_k/\lambda_{l_k} \to 0$ as $k \to \infty$;

(iv) If $\{\lambda_{n_k}\}_{k\geq 1}$ and $\{\lambda_{m_k}\}_{k\geq 1}$ are two subsequences of Λ such that $\lambda_{n_k}, \lambda_{m_k} \in G_k$ for all $k \geq 1$, then $\lambda_{n_k}/\lambda_{m_k} \to 1$ as $k \to \infty$. □

If $A \subset \mathbb{C}$ is a finite set, we will use the notation P_A for the polynomial function given by:

$$P_A(z) = \prod_{\lambda\in A} (z - \lambda). \tag{3.3}$$

With this notation, we introduce

Definition 3.5. Let $\Delta = \{G_k\}_{k\geq 1}$ be a condensation grouping of the sequence $\Lambda = \{\lambda_k\}_{k\geq 1}$. Thus,

1. The *index of condensation of* G_k is the number defined by:

$$h\left(G_k\right)=\max_{\lambda_n\in G_k}\left\{\frac{1}{\Re\left(\lambda_n\right)}\log\frac{p_k!}{\left|P'_{G_k}(\lambda_n)\right|}\right\},$$

where p_k+1 is the cardinal of the set $G_k\cap\Lambda$ and

$$\left|P'_{G_k}(\lambda_n)\right|=\prod_{\substack{\lambda_m\in G_k\\ \lambda_m\neq\lambda_n}}\left|\lambda_n-\lambda_m\right|.$$

2. The *index of the condensation grouping* $\Delta=\{G_k\}_{k\geq 1}$ is defined by:

$$h\left(\Delta\right)=\limsup h\left(G_k\right).$$

3. Finally, let us introduce the number $h(\Lambda)$ defined as follows:

$$h\left(\Lambda\right)=\sup\left\{h\left(\Delta\right):\Delta\text{ is a condensation grouping of the sequence }\Lambda\right\}.$$

Remark 3.6. Observe that if $G_k\cap\Lambda$ reduces to a single element ($p_k=0$) for some $k\geq 1$, then the function P_{G_k} is given by $P_{G_k}(z)=z-\lambda_{n_k}$ and $P'_{G_k}(z)=1$. In particular, $h\left(G_k\right)=0$. As a consequence, $h\left(\Lambda\right)\in[0,+\infty]$. □

An essential tool that will be used in the proof of our main result is the following

Theorem 3.7 ([27, Theorem 1, p. 137])**.** *Let* $\Lambda=\{\lambda_k\}_{k\geq 1}\subset\mathbb{C}$ *be a normally ordered sequence satisfying* (2.2)*. Then, for any* $q\in(0,\infty)$*, there exists a condensation grouping* $\Delta=\{G_k\}_{k\geq 1}$ *of* Λ *such that if* $\lambda_l\in G_k\cap\Lambda$ *and* $\mu_1,\ldots,\mu_n\in\Lambda$ *are points which do not belong to* G_k*, one has*

$$\prod_{j=1}^{n}\left|\lambda_l-\mu_j\right|\geq q^n n! \tag{3.4}$$

Moreover, $h\left(\Delta\right)=h\left(\Lambda\right)$.

Proof. Thanks to assumption (2.2) and using that $\{|\lambda_k|\}_{k\geq 1}$ is an increasing sequence, we deduce that the sequence $\Lambda=\{\lambda_k\}_{k\geq 1}$ has density $D=0$, i.e.,

$$D:=\lim\frac{k}{|\lambda_k|}=0.$$

Therefore, the proof of this result is exactly the same as in the cited article [27]. So we drop it. □

Theorem 3.7 was first proved by Bernstein [5] in the case of real sequences.

Our next result establishes an identity which will be crucial in the proof of the second part of Theorem 2.6. One has:

Theorem 3.8. *Let* $\Lambda=\{\lambda_k\}_{k\geq 1}\subset\mathbb{C}$ *be a normally ordered sequence satisfying condition* (2.2)*. Let us fix* $q\in(0,\infty)$ *and* $\Delta=\{G_k\}_{k\geq 1}$ *a condensation grouping of* Λ *satisfying* (3.4)*. Then, for any subsequence* $\{\lambda_{n_k}\}_{k\geq 1}\subseteq\Lambda$*, one has:*

$$\lim\left(\frac{\log\frac{1}{\left|E'(\lambda_{n_k})\right|}}{\Re(\lambda_{n_k})}-\frac{1}{\Re(\lambda_{n_k})}\log\left|\frac{q_k!}{P'_{D_k}(\lambda_{n_k})}\right|\right)=0, \tag{3.5}$$

where $\{D_k\}_{k\geq 1} \subseteq \Delta$ *is a subsequence of sets satisfying* $\lambda_{n_k} \in D_k$ *and* $q_k + 1$ *is the cardinal of the set* $D_k \cap \Lambda$*. Furthermore,*

$$c\left(\Lambda\right) = h\left(\Lambda\right),$$

where $h\left(\Lambda\right)$ *and* $c\left(\Lambda\right)$ *are respectively given in Definition 3.5 and in formula (3.2).*

Proof. Firstly, the identity $h\left(\Lambda\right) = c\left(\Lambda\right)$ can be easily deduced from (3.5) and Theorem 3.7. Therefore, let us concentrate on the proof of (3.5).

The arguments of the proof of (3.5) are inspired from those of [27, Theorem 3 and Theorem 5, p. 141]. In order to prove the result, let us fix $\{\lambda_{n_k}\}_{k\geq 1} \subset \Lambda$, a subsequence of Λ, $q \in (0,\infty)$, $\Delta = \{G_k\}_{k\geq 1}$, a condensation grouping satisfying (3.4), and $\{D_k\}_{k\geq 1} \subset \Delta$, a subsequence of Δ such that $\lambda_{n_k} \in D_k$ for all $k \geq 1$.

We introduce the notation

$$\begin{cases} P_n = \dfrac{\log \frac{1}{|E'(\lambda_n)|}}{\Re(\lambda_n)}, \quad \forall n \geq 1, \\ Q_k = \dfrac{\log \frac{1}{|E'(\lambda_{n_k})|}}{\Re(\lambda_{n_k})} - \dfrac{1}{\Re(\lambda_{n_k})} \log \left| \dfrac{q_k!}{P'_{D_k}(\lambda_{n_k})} \right|, \quad \forall k \geq 1, \end{cases}$$

where the interpolating function $E(\lambda)$ has been defined in (2.14). Then, our objective is to prove that $\lim Q_k = 0$.

Let us define the following sets:

$$A_k = \left\{ \lambda_n \notin D_k : \frac{1}{2} \leq \left| \frac{\lambda_n}{\lambda_{n_k}} \right| \leq \frac{3}{2} \right\},$$
$$B_k = \left\{ \lambda_n \in \Lambda : \left| \frac{\lambda_n}{\lambda_{n_k}} \right| < \frac{1}{2} \right\} \cup \left\{ \lambda_n \in \Lambda : \frac{3}{2} < \left| \frac{\lambda_n}{\lambda_{n_k}} \right| \right\}.$$

Observe that from the definition of condensation grouping we can deduce the existence of a positive integer k_0 such that $D_k \cap B_k = \emptyset$ for any $k \geq k_0$. So, taking into account the expression

$$\left|E'(\lambda_k)\right| = \frac{2}{|\lambda_k|} \prod_{\substack{j\geq 1 \\ j\neq k}} \left| 1 - \frac{\lambda_k^2}{\lambda_j^2} \right|, \quad \forall k \geq 1, \tag{3.6}$$

we have

$$P_{n_k} = \frac{1}{\Re\left(\lambda_{n_k}\right)} \left[\log \frac{|\lambda_{n_k}|}{2} - \left\{ \sum_{\substack{\lambda_n \in D_k \\ \lambda_n \neq \lambda_{n_k}}} + \sum_{\lambda_n \in A_k} + \sum_{\lambda_n \in B_k} \right\} \log \left| 1 - \frac{\lambda_{n_k}^2}{\lambda_n^2} \right| \right],$$

for any $k \geq k_0$.

A quick computation shows that

$$- \sum_{\substack{\lambda_n \in D_k \\ \lambda_n \neq \lambda_{n_k}}} \log \left| 1 - \frac{\lambda_{n_k}^2}{\lambda_n^2} \right| = \log \left| \frac{q_k!}{P'_{D_k}\left(\lambda_{n_k}\right)} \right| - \log \left| q_k! \prod_{\substack{\lambda_n \in D_k \\ \lambda_n \neq \lambda_{n_k}}} \frac{\lambda_n + \lambda_{n_k}}{\lambda_n^2} \right|.$$

Thus, for $k \geq k_0$,

$$\left\{ \begin{aligned} Q_k = \frac{1}{\Re\left(\lambda_{n_k}\right)} &\left[\log \frac{|\lambda_{n_k}|}{2} - \log \left| q_k! \prod_{\substack{\lambda_n \in D_k \\ \lambda_n \neq \lambda_{n_k}}} \frac{\lambda_n + \lambda_{n_k}}{\lambda_n^2} \right| \right. \\ &\left. - \left\{ \sum_{\lambda_n \in A_k} + \sum_{\lambda_n \in B_k} \right\} \log \left| 1 - \frac{\lambda_{n_k}^2}{\lambda_n^2} \right| \right] = S_k^{(1)} + S_k^{(2)} + S_k^{(3)}. \end{aligned} \right. \tag{3.7}$$

Therefore, the task will be to prove that $\lim S_k^{(i)} = 0$, for $i = 1, 2, 3$. We will divide the proof into three steps.

First step: Let us start proving that $\lim S_k^{(1)} = 0$, where

$$S_k^{(1)} = \frac{1}{\Re\left(\lambda_{n_k}\right)} \left[\log \frac{|\lambda_{n_k}|}{2} - \log \left| q_k! \prod_{\substack{\lambda_n \in D_k \\ \lambda_n \neq \lambda_{n_k}}} \frac{\lambda_n + \lambda_{n_k}}{\lambda_n^2} \right| \right].$$

Thanks to assumption (2.2) we have that $\lim |\lambda_n| = \infty$ and so,

$$\lim \frac{1}{\Re\left(\lambda_{n_k}\right)} \log \frac{|\lambda_{n_k}|}{2} = 0.$$

On the other hand, there exist $m_{k,1}$ and $m_{k,2}$ such that $\lambda_{m_{k,1}}, \lambda_{m_{k,2}} \in D_k$ and we can write

$$\left| \frac{\lambda_{m_{k,1}} + \lambda_{n_k}}{\lambda_{m_{k,1}}^2} \right| \leq \left| \frac{\lambda_n + \lambda_{n_k}}{\lambda_n^2} \right| \leq \left| \frac{\lambda_{m_{k,2}} + \lambda_{n_k}}{\lambda_{m_{k,2}}^2} \right|,$$

for any $\lambda_n \in D_k$ with $\lambda_n \neq \lambda_{n_k}$.

From Definition 3.4-(iv), for any $\lambda_n \in D_k$, there exists $\varepsilon_k^n \in \mathbb{C}$ such that $\varepsilon_k^n \to 0$ as $k \to \infty$ and $\lambda_n = \lambda_{n_k}\left(1 + \varepsilon_k^n\right)$. In particular, $\lambda_{m_{k,i}} = \lambda_{n_k}\left(1 + \varepsilon_{k,i}\right)$ with $\lim \varepsilon_{k,i} = 0$ $(i = 1, 2)$. Then, there exists $k_1 \geq k_0$ such that

$$\left\{ \begin{aligned} \left| \frac{\lambda_n + \lambda_{n_k}}{\lambda_n^2} \right| &\leq \left| \frac{\lambda_{m_{k,2}} + \lambda_{n_k}}{\lambda_{m_{k,2}}^2} \right| = \frac{1}{|\lambda_{n_k}|} \frac{|2 + \varepsilon_{k,2}|}{|1 + \varepsilon_{k,2}|^2} \leq \frac{5}{2} \frac{1}{|\lambda_{n_k}|}, \\ \left| \frac{\lambda_n + \lambda_{n_k}}{\lambda_n^2} \right| &\geq \left| \frac{\lambda_{m_{k,1}} + \lambda_{n_k}}{\lambda_{m_{k,1}}^2} \right| = \frac{1}{|\lambda_{n_k}|} \frac{|2 + \varepsilon_{k,1}|}{|1 + \varepsilon_{k,1}|^2} \geq \frac{3}{2} \frac{1}{|\lambda_{n_k}|}, \end{aligned} \right.$$

for all $\lambda_n \in D_k$ and $k \geq k_1$. Using these inequalities and Stirling's formula, we deduce that there exists a positive sequence $\{\beta_k\}_{k \geq 1}$ such that $\beta_k \to 1$ and, for $k \geq k_1$,

$$\begin{aligned} \frac{1}{\Re\left(\lambda_{n_k}\right)} \log \left(q_k! \prod_{\substack{\lambda_n \in D_k \\ \lambda_n \neq \lambda_{n_k}}} \left| \frac{\lambda_n + \lambda_{n_k}}{\lambda_n^2} \right| \right) &\geq \frac{\log \left(\beta_k \sqrt{2\pi q_k} \left(\frac{3 q_k}{2e|\lambda_{n_k}|} \right)^{q_k} \right)}{\Re\left(\lambda_{n_k}\right)} \\ &= \frac{\log \beta_k}{\Re\left(\lambda_{n_k}\right)} + \frac{1}{2} \frac{\log\left(2\pi q_k\right)}{\Re\left(\lambda_{n_k}\right)} + \frac{q_k}{\Re\left(\lambda_{n_k}\right)} \log \left(\frac{3 q_k}{2e\left|\lambda_{n_k}\right|} \right) \equiv C_{k,1} \end{aligned}$$

and

$$\frac{1}{\Re(\lambda_{n_k})}\log\left(q_k!\prod_{\substack{\lambda_n\in D_k\\ \lambda_n\neq\lambda_{n_k}}}\left|\frac{\lambda_n+\lambda_{n_k}}{\lambda_n^2}\right|\right)\leq\frac{\log\left(\beta_k\sqrt{2\pi q_k}\left(\frac{5q_k}{2e|\lambda_{n_k}|}\right)^{q_k}\right)}{\Re(\lambda_{n_k})}$$

$$=\frac{\log\beta_k}{\Re(\lambda_{n_k})}+\frac{1}{2}\frac{\log(2\pi q_k)}{\Re(\lambda_{n_k})}+\frac{q_k}{\Re(\lambda_{n_k})}\log\left(\frac{5q_k}{2e\,|\lambda_{n_k}|}\right)\equiv C_{k,2}\,.$$

Since

$$\frac{q_k}{|\lambda_{n_k}|}\leq\frac{q_k}{\Re(\lambda_{n_k})}\leq\frac{q_k}{\delta\,|\lambda_{n_k}|}$$

and q_k+1 is the cardinal of $D_k\cap\Lambda$ we get $\lim q_k/|\lambda_{n_k}|=0$ (see Definition 3.4-(iii)) , $\lim C_{k,1}=\lim C_{k,2}=0$ and

$$\lim\frac{1}{\Re(\lambda_{n_k})}\log\left|q_k!\prod_{\substack{\lambda_n\in D_k\\ \lambda_n\neq\lambda_{n_k}}}\frac{\lambda_n+\lambda_{n_k}}{\lambda_n^2}\right|=0.$$

Therefore, we have obtained $\lim S_k^{(1)}=0$.

Second step: In this step, we will deal with the element $S_k^{(2)}$ in (3.7):

$$S_k^{(2)}=-\frac{1}{\Re(\lambda_{n_k})}\sum_{\lambda_n\in A_k}\log\left|1-\frac{\lambda_{n_k}^2}{\lambda_n^2}\right|.$$

Let α_k be the cardinal number of A_k. Since for $\lambda_n\in A_k$ we have $|\lambda_n|\leq\frac{3}{2}|\lambda_{n_k}|$, we deduce

$$\alpha_k\leq\mathcal{N}\left(\frac{3}{2}|\lambda_{n_k}|\right),$$

where, for every real number x, $\mathcal{N}(x)$ is the cardinal of the set $\{n\in\mathbb{N}:|\lambda_n|<x\}$.

Observe that $\sum_{n\geq1}1/|\lambda_n|<\infty$ and $\{1/|\lambda_n|\}_{n\geq1}$ is a non-increasing sequence. We easily deduce that $n/|\lambda_n|\to0$ as $n\to\infty$ and thus

$$\lim_{x\to\infty}\frac{\mathcal{N}(x)}{x}=0. \tag{3.8}$$

So, it follows that for any $\varepsilon>0$, there exists $R(\varepsilon)>0$ such that $0\leq\mathcal{N}(x)\leq\varepsilon x$ for any $x\geq R$. In particular, there is $k_2(\varepsilon)\in\mathbb{N}$ such that

$$0\leq\alpha_k\leq\frac{3}{2}\varepsilon\,|\lambda_{n_k}|\leq\frac{3}{2\delta}\varepsilon\,\Re(\lambda_{n_k}),\quad\forall k\geq k_2\,.$$

This last inequality proves

$$\lim\frac{\alpha_k}{\Re(\lambda_{n_k})}=0. \tag{3.9}$$

Observe that the sequence Λ satisfies $0<\delta\,|\lambda_n|\leq\Re(\lambda_n)$ for all $n\geq1$ (see (2.2)). In particular, there exists $\phi_\delta\in[0,\pi/2)$ such that $|\arg(\lambda_n)|\leq\phi_\delta<\pi/2$ for any $n\geq1$. Let us introduce the function $g:z\in\mathbb{C}\mapsto g(z)=z(z+1)$ and the compact set

$$\mathcal{O}_\delta:=\left\{z\in\mathbb{C}:\frac{2}{3}\leq|z|\leq2,\quad|\arg z|\leq2\phi_\delta<\pi\right\}. \tag{3.10}$$

Now, using the definition of A_k, we deduce

$$\frac{\lambda_{n_k}}{\lambda_n} \in \mathcal{O}_\delta, \quad \forall k \geq 1, \quad \forall \lambda_n \in A_k.$$

Therefore, we can bound

$$\left|\frac{\lambda_n + \lambda_{n_k}}{\lambda_n^2}\right| = \frac{1}{|\lambda_{n_k}|} |g(\lambda_{n_k}/\lambda_n)| \geq \frac{a}{|\lambda_{n_k}|}, \quad \forall k \geq 1, \quad \forall \lambda_n \in A_k, \tag{3.11}$$

with $a = \min_{z \in \mathcal{O}_\delta} |g(z)| \in (0, \infty)$.

Recall that the condensation grouping $\Delta = \{G_k\}_{k \geq 1}$ satisfies (3.4) for $q \in (0, \infty)$. In particular,

$$\prod_{\lambda_n \in A_k} |\lambda_n - \lambda_{n_k}| \geq q^{\alpha_k} \alpha_k!, \quad \forall k \geq 1,$$

where α_k is the cardinal of the set A_k.

From this last inequality, inequality (3.11) and again Stirling's formula, it follows that there exists a sequence $\{\beta_k\}_{k \geq 1}$ satisfying $\lim \beta_k = 1$ for which

$$\begin{aligned}
\sum_{\lambda_n \in A_k} \log \left|1 - \frac{\lambda_{n_k}^2}{\lambda_n^2}\right| &= \log \prod_{\lambda_n \in A_k} |\lambda_n - \lambda_{n_k}| + \log \prod_{\lambda_n \in A_k} \left|\frac{\lambda_n + \lambda_{n_k}}{\lambda_n^2}\right| \\
&\geq \log\left(q^{\alpha_k} \alpha_k!\right) + \log\left(\frac{a}{|\lambda_{n_k}|}\right)^{\alpha_k} \\
&\geq \log\left(\beta_k \sqrt{2\pi\alpha_k} \left(\frac{a\delta q \alpha_k}{e\Re(\lambda_{n_k})}\right)^{\alpha_k}\right),
\end{aligned}$$

for any $k \geq 1$. Therefore,

$$\left\{\begin{aligned}
S_k^{(2)} &\leq -\frac{1}{\Re(\lambda_{n_k})} \log\left(\beta_k \sqrt{2\pi\alpha_k} \left(\frac{a\delta q \alpha_k}{e\Re(\lambda_{n_k})}\right)^{\alpha_k}\right) \\
&= -\frac{\log(\beta_k)}{\Re(\lambda_{n_k})} - \frac{\log\sqrt{2\pi\alpha_k}}{\Re(\lambda_{n_k})} - \frac{\alpha_k}{\Re(\lambda_{n_k})} \log\left(\frac{C\alpha_k}{\Re(\lambda_{n_k})}\right) := \Gamma_{k,1},
\end{aligned}\right.$$

for all $k \geq 1$.

Finally, using the definition of A_k, we get

$$S_k^{(2)} \geq -\frac{1}{\Re(\lambda_{n_k})} \sum_{\lambda_n \in A_k} \log\left(1 + \left|\frac{\lambda_{n_k}^2}{\lambda_n^2}\right|\right) \geq -\frac{\alpha_k}{\Re(\lambda_{n_k})} \log 5 := \Gamma_{k,2},$$

for any $k \geq 1$.

Observe that (3.9) implies $\lim \Gamma_{k,1} = \lim \Gamma_{k,2} = 0$. Thus, the two previous inequalities directly provide $\lim S_k^{(2)} = 0$.

Third step: In this last step we will prove $\lim S_k^{(3)} = 0$. Let us recall (see (3.7)) that

$$S_k^{(3)} = -\frac{1}{\Re(\lambda_{n_k})} \sum_{\lambda_n \in B_k} \log\left|1 - \frac{\lambda_{n_k}^2}{\lambda_n^2}\right|.$$

It is easy to see the inequalities

$$\sum_{\lambda_n \in B_k} \log \left| 1 - \frac{|\lambda_{n_k}|^2}{|\lambda_n|^2} \right| \leq \sum_{\lambda_n \in B_k} \log \left| 1 - \frac{\lambda_{n_k}^2}{\lambda_n^2} \right| \leq \sum_{\lambda_n \in B_k} \log \left(1 + \frac{|\lambda_{n_k}|^2}{|\lambda_n|^2} \right).$$

Therefore,

$$-\frac{1}{\Re\left(\lambda_{n_k}\right)} \sum_{\lambda_n \in B_k} f_+(|\lambda_n|) \leq S_k^{(3)} \leq -\frac{1}{\Re\left(\lambda_{n_k}\right)} \sum_{\lambda_n \in B_k} f_-(|\lambda_n|), \quad \forall k \geq 1,$$

where,

$$f_\pm(x) = \log \left| 1 \pm \frac{|\lambda_{n_k}|^2}{x^2} \right|, \quad \text{with } x > 0.$$

Thus, in order to obtain $\lim S_k^{(3)} = 0$, it is enough to see that

$$\lim \left(\frac{1}{\Re\left(\lambda_{n_k}\right)} \sum_{\lambda_n \in B_k} f_\pm(|\lambda_n|) \right) = 0. \tag{3.12}$$

Indeed, we have

$$\begin{aligned} \sum_{\lambda_n \in B_k} f_\pm(|\lambda_n|) &= \left(\int_{|\lambda_1|}^{\frac{|\lambda_{n_k}|}{2}} + \int_{\frac{3}{2}|\lambda_{n_k}|}^{\infty} \right) f_\pm(x)\, d\mathcal{N}(x) \\ &= \left[\mathcal{N}(x)\, f_\pm(x)\right]_{|\lambda_1|}^{\frac{1}{2}|\lambda_{n_k}|} + \left[\mathcal{N}(x)\, f_\pm(x)\right]_{\frac{3}{2}|\lambda_{n_k}|}^{\infty} \\ &\quad - \left(\int_{|\lambda_1|}^{\frac{1}{2}|\lambda_{n_k}|} + \int_{\frac{3}{2}|\lambda_{n_k}|}^{\infty} \right) f_\pm'(x) \mathcal{N}(x)\ dx = \sum_{i=1}^{4} I_{k,i}\,, \end{aligned} \tag{3.13}$$

where we recall that $\mathcal{N}(x)$ gives, for each $x > 0$, the cardinal of the set $\{n \in \mathbb{N} : |\lambda_n| < x\}$ and satisfies (3.8).

First, observe that $\mathcal{N}(|\lambda_1|) = 0$. Secondly, from (3.8), we can also check

$$\lim_{x\to\infty} \left(\mathcal{N}(x)\, f_\pm(x)\right) = 0.$$

Thus, the first two terms in (3.13) can be evaluated as follows:

$$I_{k,1} = \log|1 \pm 4|\, \mathcal{N}\left(\frac{1}{2}|\lambda_{n_k}|\right), \quad I_{k,2} = -\log\left|1 \pm \frac{4}{9}\right| \mathcal{N}\left(\frac{3}{2}|\lambda_{n_k}|\right).$$

Using again (3.8) we get

$$\lim \frac{1}{\Re\left(\lambda_{n_k}\right)} \left(I_{k,1} + I_{k,2}\right) = 0. \tag{3.14}$$

Now, we will deal with the term $I_{k,3}$ in (3.13). Using once more (3.8) and also (2.2) we obtain that, for any $\varepsilon > 0$, there exists $k_3 \geq 1$ such that, if $k \geq k_3$, one has

$$\begin{cases} 0 < \dfrac{\mathcal{N}(x)}{x} \leq \varepsilon, \quad \forall x > \sqrt{|\lambda_{n_k}|}, \quad \text{and} \\ \displaystyle\int_{|\lambda_1|/|\lambda_{n_k}|}^{1/\sqrt{|\lambda_{n_k}|}} \frac{1}{|x^2 \pm 1|}\, dx \leq \varepsilon. \end{cases}$$

So, for $k \geq k_3$, we can write:

$$\begin{aligned}
|I_{k,3}| &= \left| \left(\int_{|\lambda_1|}^{\sqrt{|\lambda_{n_k}|}} + \int_{\sqrt{|\lambda_{n_k}|}}^{\frac{1}{2}|\lambda_{n_k}|} \right) f'_{\pm}(x)\mathcal{N}(x)\, dx \right| \\
&= 2\,|\lambda_{n_k}|^2 \left| \left(\int_{|\lambda_1|}^{\sqrt{|\lambda_{n_k}|}} + \int_{\sqrt{|\lambda_{n_k}|}}^{\frac{1}{2}|\lambda_{n_k}|} \right) \frac{\mathcal{N}(x)}{x} \frac{1}{x^2 \pm |\lambda_{n_k}|^2}\, dx \right| \\
&\leq 2\,|\lambda_{n_k}|^2 \left(\int_{|\lambda_1|}^{\sqrt{|\lambda_{n_k}|}} \frac{M}{\left|x^2 \pm |\lambda_{n_k}|^2\right|}\, dx + \int_{\sqrt{|\lambda_{n_k}|}}^{|\lambda_{n_k}|/2} \frac{\varepsilon}{\left|x^2 \pm |\lambda_{n_k}|^2\right|}\, dx \right) \\
&\leq 2\,|\lambda_{n_k}| \left(M \int_{|\lambda_1|/|\lambda_{n_k}|}^{1/\sqrt{|\lambda_{n_k}|}} \frac{1}{|x^2 \pm 1|}\, dx + \varepsilon \int_{1/\sqrt{|\lambda_{n_k}|}}^{1/2} \frac{1}{|x^2 \pm 1|}\, dx \right) \\
&\leq C\,|\lambda_{n_k}|\,\varepsilon.
\end{aligned}$$

Thus since $\varepsilon > 0$ is arbitrary, the previous inequality gives

$$\lim \frac{I_{k,3}}{\Re(\lambda_{n_k})} = 0. \tag{3.15}$$

Finally, let us work with the last term $I_{k,4}$ in (3.13). Using again (3.8) and reasoning as before, we can see that for arbitrary $\varepsilon > 0$, there exists a large k_4 such that the following inequality

$$\begin{aligned}
|I_{k,4}| &= 2\,|\lambda_{n_k}|^2 \left| \int_{3|\lambda_{n_k}|/2}^{\infty} \frac{\mathcal{N}(x)}{x} \frac{1}{x^2 \pm |\lambda_{n_k}|^2} dx \right| \\
&\leq 2\,|\lambda_{n_k}|\,\varepsilon \left| \int_{3/2}^{\infty} \frac{1}{x^2 \pm 1} dx \right|
\end{aligned}$$

holds for any $k \geq k_4$. Thus again

$$\lim \frac{I_{k,4}}{\Re(\lambda_{n_k})} = 0. \tag{3.16}$$

From (3.13) and using (3.14)–(3.16), we have (3.12) and $\lim S_k^{(3)} = 0$.

Going back to the expression (3.7), the previous steps prove (3.5). This finalizes the proof. □

Theorem 3.8 provides a very important identity (see (3.5)). In particular, this identity allows us to deduce a first formula for calculating the condensation index $c(\Lambda)$ of a complex sequence $\Lambda = \{\lambda_k\}_{k\geq 1} \subset \mathbb{C}$ satisfying (2.2). In the next result we are going to obtain a new formula for the condensation index which relates $c(\Lambda)$ to the Blaschke product associated with the sequence Λ.

Let $\Lambda = \{\lambda_k\}_{k\geq 1} \subset \mathbb{C}_+$ be a complex sequence satisfying $\lambda_k \neq \lambda_i$, for any $i \neq j$, and

$$\sum_{k\geq 1} \frac{\Re(\lambda_k)}{[1 + \Re(\lambda_k)]^2 + [\Im(\lambda_k)]^2} < \infty. \tag{3.17}$$

Then, we recall that the Blaschke product associated with this sequence is the function $W : \mathbb{C}_+ \to \mathbb{C}$ defined by:

$$\begin{cases} W(\lambda) = W(\lambda, \Lambda) = \displaystyle\prod_{k\geq 1} \delta_k \frac{1 - \lambda/\lambda_k}{1 + \lambda/\overline{\lambda}_k}, \quad \lambda \in \mathbb{C}_+, \\ \delta_k = \dfrac{\lambda_k}{\overline{\lambda}_k} \dfrac{|\lambda_k - 1|}{|\lambda_k + 1|} \dfrac{\overline{\lambda}_k + 1}{\overline{\lambda}_k - 1} \quad (\delta_k = 1 \text{ if } \lambda_k = 1). \end{cases} \tag{3.18}$$

We also recall (see for instance [3]) that, under assumption (3.17), the Blaschke product satisfies $W \in H^\infty(\mathbb{C}_+)$, the space of bounded and holomorphic functions defined on $\mathbb{C}_+$, is defined almost everywhere on $i\mathbb{R}$ and satisfies $|W(\lambda)| < 1$, for $\Re\lambda > 0$, and $|W(i\tau)| = 1$, for almost every $\tau \in \mathbb{R}$.

One has:

Theorem 3.9. *Let $\Lambda = \{\lambda_k\}_{k\geq 1} \subset \mathbb{C}$ be a normally ordered sequence satisfying condition (2.2). Then, for any subsequence $\{\lambda_{n_k}\}_{k\geq 1} \subseteq \Lambda$, the following identity holds:*

$$\lim \left(\frac{\log \frac{1}{|E'(\lambda_{n_k})|}}{\Re(\lambda_{n_k})} - \frac{\log \frac{1}{|W'(\lambda_{n_k})|}}{\Re(\lambda_{n_k})} \right) = 0.$$

In particular,

$$c\,(\Lambda) = \limsup \frac{\log \frac{1}{|W'(\lambda_k)|}}{\Re\,(\lambda_k)},$$

where $c\,(\Lambda)$ and the function $W(\lambda)$ are respectively given in formulas (3.2) and (3.18).

Proof. Let us consider a sequence $\Lambda = \{\lambda_k\}_{k\geq 1}$ satisfying (2.2). In particular, the sequence Λ also satisfies (3.17) and then, the Blaschke function $W(\lambda)$ is well-defined on $\mathbb{C}_+$. In fact, we can readily check the formula

$$W'(\lambda_k) = -\delta_k \frac{-\overline{\lambda}_k}{2\lambda_k \Re(\lambda_k)} \prod_{\substack{j\geq 1\\ j\neq k}} \delta_j \frac{1 - \lambda_k/\lambda_j}{1 + \lambda_k/\overline{\lambda}_j}\,. \tag{3.19}$$

The objective of the proof is to show that $\lim \mathcal{Q}_{n_k} = 0$, where

$$\mathcal{Q}_{n_k} = \frac{\log \frac{1}{|E'(\lambda_{n_k})|}}{\Re(\lambda_{n_k})} - \frac{\log \frac{1}{|W'(\lambda_{n_k})|}}{\Re(\lambda_{n_k})}.$$

Evidently, this property provides the new expression for $c\,(\Lambda)$, the condensation index of the sequence $\Lambda = \{\lambda_k\}_{k\geq 1}$.

Thanks to the expression of $|E'(\lambda_{n_k})|$ (see (3.6)), we can readily calculate $\mathcal{Q}_{n_k}$ and obtain

$$\left\{ \begin{aligned} \mathcal{Q}_{n_k} &= \frac{1}{\Re(\lambda_{n_k})} \log \left[\frac{|\lambda_{n_k}|}{4\Re(\lambda_{n_k})} \prod_{\substack{j\geq 1\\ j\neq n_k}} \frac{1}{\left|1 + \frac{\lambda_{n_k}}{\lambda_j}\right| \left|1 + \frac{\lambda_{n_k}}{\overline{\lambda}_j}\right|} \right] \\ &= \frac{1}{\Re(\lambda_{n_k})} \log \left[\frac{1}{2} \prod_{j\geq 1} \frac{1}{\left|1 + \frac{\lambda_{n_k}}{\lambda_j}\right| \left|1 + \frac{\lambda_{n_k}}{\overline{\lambda}_j}\right|} \right]. \end{aligned} \right. \tag{3.20}$$

Let us first prove that

$$\liminf \mathcal{Q}_{n_k} \geq 0. \tag{3.21}$$

To this end, let us fix $\varepsilon > 0$. Using assumption (2.2), we deduce the existence of $N_0(\varepsilon) \in \mathbb{N}$ such that

$$\sum_{j > N_0(\varepsilon)} \frac{1}{|\lambda_j|} \leq \frac{\varepsilon}{2}.$$

Therefore, we can bound

$$\prod_{j\geq 1}\left|1+\frac{\lambda_{n_k}}{\lambda_j}\right|\left|1+\frac{\lambda_{n_k}}{\overline{\lambda}_j}\right|\leq \prod_{j\geq 1}\left(1+\frac{|\lambda_{n_k}|}{|\lambda_j|}\right)^2=\prod_{j=1}^{N_0(\varepsilon)}\left(1+\frac{|\lambda_{n_k}|}{|\lambda_j|}\right)^2\prod_{j>N_0(\varepsilon)}\left(1+\frac{|\lambda_{n_k}|}{|\lambda_j|}\right)^2$$
$$\leq\left(1+\frac{|\lambda_{n_k}|}{|\lambda_1|}\right)^{2N_0(\varepsilon)}\prod_{j>N_0(\varepsilon)}e^{2|\lambda_{n_k}|/|\lambda_j|}\leq\left(1+\frac{|\lambda_{n_k}|}{|\lambda_1|}\right)^{2N_0(\varepsilon)}e^{\varepsilon|\lambda_{n_k}|}.$$

In the previous estimate we have used the inequality $1+x\leq e^x$ valid for any $x\geq 0$. We have also used that the sequence Λ is normally ordered and, therefore, $|\lambda_j|\geq|\lambda_1|$ for any $j\geq 1$. From the previous inequality and using again (2.2), we also obtain

$$\mathcal{Q}_{n_k}\geq\frac{1}{\Re(\lambda_{n_k})}\log\left[\frac{1}{2}\left(1+\frac{|\lambda_{n_k}|}{|\lambda_1|}\right)^{-2N_0(\varepsilon)}e^{-\varepsilon|\lambda_{n_k}|}\right]\geq\frac{-\log 2-2N_0(\varepsilon)\log\left(1+\frac{|\lambda_{n_k}|}{|\lambda_1|}\right)}{\Re(\lambda_{n_k})}-\frac{\varepsilon}{\delta}.$$

From this inequality we get $\liminf \mathcal{Q}_{n_k}\geq-\varepsilon/\delta$, for any positive ε. Thus, we have proved (3.21).

Let us now see the inequality

$$\limsup \mathcal{Q}_{n_k}\leq 0. \tag{3.22}$$

To this end we will use the sets:

$$\widetilde{A}_k^{(1)}=\left\{j\geq 1:\frac{1}{2}\leq\frac{|\lambda_j|}{|\lambda_{n_k}|}\leq\frac{3}{2}\right\},$$
$$\widetilde{A}_k^{(2)}=\left\{j:\frac{|\lambda_j|}{|\lambda_{n_k}|}<\frac{1}{2}\right\},\quad \widetilde{A}_k^{(3)}=\left\{j:\frac{3}{2}<\frac{|\lambda_j|}{|\lambda_{n_k}|}\right\}.$$

We can decompose the infinite product in (3.20) into three product $\Pi_k^{(1)}$, $\Pi_k^{(2)}$ and $\Pi_k^{(3)}$, where:

$$\Pi_k^{(i)}=\prod_{j\in\widetilde{A}_k^{(i)}}\frac{1}{\left|1+\frac{\lambda_{n_k}}{\lambda_j}\right|\left|1+\frac{\lambda_{n_k}}{\overline{\lambda}_j}\right|},\quad\forall k\geq 1,\quad\text{with } i=1,2,3. \tag{3.23}$$

First product: Following the ideas of the proof of Theorem 3.8, let $\widetilde{\alpha}_k$ be the cardinal of the set $\widetilde{A}_k^{(1)}$. Then,

$$\widetilde{\alpha}_k\leq\mathcal{N}\left(\frac{3}{2}|\lambda_{n_k}|\right),$$

where, for every real number x, $\mathcal{N}(x)$ is the cardinal of the set $\{n\in\mathbb{N}:|\lambda_n|<x\}$. Using once more (3.8), we can also deduce

$$\lim\frac{\widetilde{\alpha}_k}{\Re\left(\lambda_{n_k}\right)}=0. \tag{3.24}$$

Let us also consider the compact set $\mathcal{O}_\delta$ given by (3.10), where $\phi_\delta\in[0,\pi/2)$ is such that $\arg\left(\lambda_i\right)\leq\phi_\delta$ for any $i\geq 1$. It is clear that

$$\frac{\lambda_{n_k}}{\lambda_j},\frac{\lambda_{n_k}}{\overline{\lambda}_j}\in\mathcal{O}_\delta,\quad\forall k\geq 1,\quad\forall j\in\widetilde{A}_k^{(1)}.$$

and thus

$$\Pi_k^{(1)}=\prod_{j\in\widetilde{A}_k^{(1)}}\left|h(\lambda_{n_k}/\lambda_j)\right|\left|h(\lambda_{n_k}/\overline{\lambda}_j)\right|\leq b^{2\widetilde{\alpha}_k},$$

where $h(z) = (1+z)^{-1}$ and $b = \sup_{z\in\mathcal{O}_\delta} |h(z)| \in (0,\infty)$. Using (3.24) and the previous inequality we infer

$$\limsup \frac{1}{\Re(\lambda_{n_k})} \log \Pi_k^{(1)} \le \lim \left(\frac{2\widetilde{\alpha}_k}{\Re(\lambda_{n_k})} \log b \right) = 0. \tag{3.25}$$

Second product: Let us now consider $\Pi_k^{(2)}$ (see (3.23)). From the definition of the set $\widetilde{A}_k^{(2)}$, we can directly bound

$$\Pi_k^{(2)} \le \prod_{j\in\widetilde{A}_k^{(2)}} \frac{1}{\left(\frac{|\lambda_{n_k}|}{|\lambda_j|} - 1\right)\left(\frac{|\lambda_{n_k}|}{|\overline{\lambda}_j|} - 1\right)} \le 1,$$

whence

$$\limsup \frac{1}{\Re(\lambda_{n_k})} \log \Pi_k^{(2)} \le 0. \tag{3.26}$$

Third product: Finally, let us take the third product $\Pi_k^{(3)}$. Using the definition of the set $\widetilde{A}_k^{(3)}$, we get

$$\Pi_k^{(3)} \le \prod_{j\in\widetilde{A}_k^{(3)}} \frac{1}{\left(1 - \frac{|\lambda_{n_k}|}{|\lambda_j|}\right)\left(1 - \frac{|\lambda_{n_k}|}{|\overline{\lambda}_j|}\right)} \le \prod_{j\in\widetilde{A}_k^{(3)}} e^{2|\lambda_{n_k}|/|\lambda_j|} e^{2|\lambda_{n_k}|/|\overline{\lambda}_j|}.$$

In this inequality we have used the inequality $1 - x \ge e^{-2x}$ which is valid for any $x \in [0, 2/3]$. So,

$$\limsup \frac{1}{\Re(\lambda_{n_k})} \log \Pi_k^{(3)} \le \limsup \left(\frac{4|\lambda_{n_k}|}{\Re(\lambda_{n_k})} \sum_{j\in\widetilde{A}_k^{(3)}} \frac{1}{|\lambda_j|} \right) = 0. \tag{3.27}$$

Coming back to the formula of $\mathcal{Q}_{n_k}$ (see (3.20)), from inequalities (3.25)–(3.27) we obtain (3.22). If we now add property (3.21), we get $\lim \mathcal{Q}_{n_k} = 0$ and the proof of the result. □

Remark 3.10. Theorems 3.8 and 3.9 provides two different formulas for calculating the condensation index of a complex sequence. The second one will be used for proving the first part of the main result Theorem 2.6 and the first one for the proof of the second part of this result. Observe that from Remark 3.6 and the identity $c(\Lambda) = h(\Lambda)$ we deduce that $c(\Lambda) \in [0,\infty]$ for any complex sequence $\Lambda = \{\lambda_k\}_{k\ge1}$ satisfying (2.2). In Section 6, we will provide examples of sequences Λ for which $c(\Lambda)$ could have any value in the interval $[0,\infty]$. □

We will finish this section recalling some conditions on the sequence Λ which ensure that the corresponding index of condensation is zero. One has:

Proposition 3.11. *Let $\Lambda = \{\lambda_k\}_{k\ge1} \subset \mathbb{C}$ be a sequence satisfying condition (2.2). Let us also assume that the sequence Λ satisfies one of the following conditions*

1. *There exist a positive constant $\rho > 0$ and a positive integer n_0 such that*

$$|\lambda_k - \lambda_l| \ge \rho|k-l|, \quad \forall k, l \ge n_0.$$

2. *There exist a positive constant $\widetilde{\rho} > 0$ and a positive integer n_1 such that*

$$|\lambda_k - \lambda_l| \ge \widetilde{\rho}|\lambda_k|^{1/2}, \quad \forall k \ge n_1 \text{ and } l \ne k.$$

Then, $c(\Lambda) = 0$. □

For a proof of this proposition in the real or complex cases, see for instance [5, 10, 11, 27, 15, 12, 3].

Remark 3.12. Observe that in the case of the boundary null controllability of a single one-dimensional heat equation (see [10] and [11]) we are precisely in the first case of Proposition 3.11. Indeed, in this case $\Lambda = \{\mu_k\}_{k\geq 1}$ (see (1.3)) which, evidently, satisfies the first point of Proposition 3.11. So, applying Theorem 2.6, we will see that the null controllability result for the heat equation with boundary controls is valid for every $T > 0$ (see Remark 6.27). □

4 Existence of biorthogonal families to complex exponentials. Some properties of the minimal controllability time

In this section we will give a result on existence of biorthogonal families to complex exponentials. In addition, we will study some properties of these families. As a consequence we will give some properties of the minimal controllability time T_0 given by formula (2.13).

Let us start studying a result on existence of a biorthogonal family in $L^2(0,T;\mathbb{C})$ to the complex exponential sequence $\{e^{-\lambda_k t}\}_{k\geq 1}$, where $\{\lambda_k\}_{k\geq 1}$ is a complex sequence satisfying appropriate properties.

Given $T \in (0,\infty]$ and $\Lambda = \{\lambda_k\}_{k\geq 1} \subset \mathbb{C}_+$ a complex sequence, let us consider the closed space $A(\Lambda,T) \subset L^2(0,T;\mathbb{C})$ given by

$$A(\Lambda,T) = \overline{\text{span }\{e^{-\lambda_k t} : k \geq 1\}}^{L^2(0,T;\mathbb{C})}.$$

In the sequel, $H^2(\mathbb{C}_+)$ will denote the Hardy space of holomorphic functions Φ on $\mathbb{C}_+$ such that

$$\int_{-\infty}^{+\infty} |\Phi(\sigma + i\tau)|^2 \, d\tau < \infty, \quad \forall \sigma > 0,$$

with norm

$$\|\Phi\|_{H^2(\mathbb{C}_+)} = \left(\int_{-\infty}^{+\infty} |\Phi\,(i\tau)|^2 \; d\tau \right)^{1/2}.$$

(For the space $H^2(\mathbb{C}_+)$ and the properties of the Laplace transform, see for instance [28, pp. 19–20]).

Let us also consider the function

$$J(\lambda) = \frac{W(\lambda)}{(1+\lambda)^2}, \quad \text{for } \lambda \in \mathbb{C}_+\,, \tag{4.1}$$

where W is the infinite Blaschke product given by (3.18).

With the previous notation, one has the following result:

Theorem 4.1. *Let* $\Lambda = \{\lambda_k\}_{k\geq 1} \subset \mathbb{C}$ *be a sequence satisfying* (2.2) *and fix* $T \in (0,\infty]$*. Then, there exists a biorthogonal family* $\{q_k\}_{k\geq 1} \subset A(\Lambda,T)$ *to* $\left\{e^{-\lambda_k t}\right\}_{k\geq 1}$ *such that*

$$C_1 \frac{\|J\|_{H^2(\mathbb{C}_+)}}{|\lambda_k||W'(\lambda_k)|}|1+\lambda_k|^2 \leq \|q_k\|_{L^2(0,T;\mathbb{C})} \leq C_2 \frac{\|J\|_{H^2(\mathbb{C}_+)}}{|\lambda_k||W'(\lambda_k)|}|1+\lambda_k|^2, \quad \forall k \geq 1, \tag{4.2}$$

where C_1 and C_2 are positive constants only depending on Λ and T and W is the function given by (3.18)*. Furthermore, for any $\varepsilon > 0$ one has*

$$C_{1,\varepsilon}\frac{e^{-\varepsilon\Re(\lambda_k)}}{|E'(\lambda_k)|} \le \|q_k\|_{L^2(0,T;\mathbb{C})} \le C_{2,\varepsilon}\frac{e^{\varepsilon\Re(\lambda_k)}}{|E'(\lambda_k)|}, \quad \forall k \ge 1, \tag{4.3}$$

where E is the function given in (2.14) *and $C_{1,\varepsilon}, C_{2,\varepsilon} > 0$ are constants only depending on ε, Λ and T.*

Proof. Let us take a sequence $\Lambda = \{\lambda_k\}_{k\ge1} \subset \mathbb{C}$ satisfying (2.2) and let us first work in the case $T = \infty$. Observe that, in particular, the sequence Λ satisfies (3.17) and this condition guarantees that $A(\Lambda,\infty)$ is a proper closed subspace of $L^2(0,\infty;\mathbb{C})$ (see [3]). In fact this condition also ensures the existence of a biorthogonal family $\{\widetilde{q}_k\}_{k\ge1} \subset A(\Lambda,\infty)$ to the exponentials $\{e^{-\lambda_k t}\}_{k\ge1}$ in $L^2(0,\infty;\mathbb{C})$ (see for instance [3, Proposition 4.1]). In order to prove inequalities (4.2) and (4.3), let us recall how the biorthogonal family $\{\widetilde{q}_k\}_{k\ge1}$ can be obtained (for the details, see [3]).

Let us consider the function

$$J_k(\lambda) := \frac{J(\lambda)}{J'(\lambda_k)(\lambda-\lambda_k)}, \quad \text{for } \lambda \in \mathbb{C}_+ ,$$

where J is given in (4.1). Simple computations immediately show that $J, J_k \in H^2(\mathbb{C}_+)$, for any $k \ge 1$, and

$$J_k(\lambda_l) = \delta_{kl}, \quad \forall k,l \ge 1.$$

So, using that the Laplace transform is a homeomorphism from $L^2(0,\infty;\mathbb{C})$ into $H^2(\mathbb{C}_+)$, we infer the existence of a nontrivial function $\widetilde{q}_k \in L^2(0,\infty;\mathbb{C})$ such that

$$J_k(\lambda) := \frac{J(\lambda)}{J'(\lambda_k)(\lambda-\lambda_k)} = \int_0^\infty e^{-\lambda t}\overline{\widetilde{q}_k}(t)\,dt, \quad \forall\lambda \in \mathbb{C}_+ . \tag{4.4}$$

Using the equalities $J_k(\lambda_l) = \delta_{kl}$, we get

$$\int_0^\infty e^{-\lambda_l t}\overline{\widetilde{q}_k}(t)\,dt = \delta_{kl}, \quad \forall k,l \ge 1,$$

i.e., $\{\widetilde{q}_k\}_{k\ge1}$ is a biorthogonal family to $\{e^{-\lambda_k t}\}_{k\ge1}$ in $L^2(0,\infty;\mathbb{C})$.

The Parseval equality gives

$$\frac{1}{2\pi}\|\widetilde{q}_k\|^2_{L^2(0,\infty;\mathbb{C})} = \int_{-\infty}^{+\infty} |J_k(i\tau)|^2\,d\tau = \frac{1}{|J'(\lambda_k)|^2}\int_{-\infty}^{+\infty}\frac{|J(i\tau)|^2}{|i\tau-\lambda_k|^2}\,d\tau, \quad \forall k \ge 1,$$

whence

$$\begin{aligned}\lim\left(\frac{1}{2\pi}|\lambda_k J'(\lambda_k)|^2\|\widetilde{q}_k\|^2_{L^2(0,\infty;\mathbb{C})}\right) &= \lim\int_{-\infty}^{+\infty}\frac{|\lambda_k|^2|J(i\tau)|^2}{|i\tau-\lambda_k|^2}\,d\tau\\ &= \int_{-\infty}^{+\infty}|J(i\tau)|^2\,d\tau = \|J\|^2_{H^2(\mathbb{C}_+)}.\end{aligned}$$

The previous equality can be deduced as a direct consequence of the Lebesgue's dominated convergence theorem. This proves inequality (4.2) in the case $T = \infty$.

Let us now prove inequality (4.3) in the case $T = \infty$. Given $\varepsilon > 0$, from Theorem 3.9 is not difficult to prove the existence of $k_0(\varepsilon) \in \mathbb{N}$ such that

$$e^{-\frac{\varepsilon}{2}\Re(\lambda_k)}\frac{1}{|E'(\lambda_k)|} \le \frac{1}{|W'(\lambda_k)|} \le e^{\frac{\varepsilon}{2}\Re(\lambda_k)}\frac{1}{|E'(\lambda_k)|}, \quad \forall k \ge k_0(\varepsilon).$$

On the other hand, using assumption (2.2) we also deduce that there exists $k_1(\varepsilon) \in \mathbb{N}$ such that

$$1 \leq \frac{|1+\lambda_k|^2}{|\lambda_k|} \leq e^{\frac{\varepsilon}{2}\Re(\lambda_k)}, \quad \forall k \geq k_1(\varepsilon).$$

These two inequalities together with (4.2) provide inequality (4.3) in the case $T = \infty$. This completely proves the result in the case $T = \infty$.

The general case $T \in (0, \infty)$ will be deduced from the following

Lemma 4.2. *Let $\Lambda = \{\lambda_k\}_{k\geq 1} \subset \mathbb{C}$ be a sequence satisfying (2.2). Then, for any $T \in (0,\infty)$, the restriction operator*

$$R_T : \varphi \in A(\Lambda, \infty) \longmapsto R_T\varphi = \varphi_{|(0,T)} \in A(\Lambda, T)$$

is an isomorphism. In particular, there exists a positive constant C_T, depending on the sequence Λ and T, such that

$$\|\varphi\|_{L^2(0,\infty;\mathbb{C})} \leq C_T\|\varphi\|_{L^2(0,T;\mathbb{C})}, \quad \forall \varphi \in A(\Lambda, \infty). \qquad \square$$

Before proving Lemma 4.2, let us complete the proof of Theorem 4.1 in the case $T \in (0, \infty)$. Applying Theorem 4.1 in the case $T = \infty$, we deduce the existence of a family $\{\widetilde{q}_k\}_{k\geq 1} \subset A(\Lambda, \infty)$ biorthogonal to $\left\{e^{-\lambda_k t}\right\}_{k\geq 1}$ in $L^2(0, \infty; \mathbb{C})$ which satisfies (4.2) and (4.3).

Let us set

$$q_k = \left(R_T^{-1}\right)^* \widetilde{q}_k \in A(\Lambda, T), \quad \forall k \geq 1.$$

From Lemma 4.2 and the properties of the family $\{\widetilde{q}_k\}_{k\geq 1}$, it is clear that the function q_k satisfies for any $k \geq 1$ inequalities (4.2) and (4.3) (this last inequality for any $\varepsilon > 0$).

On the other hand, with the notation $\varphi_k(t) = e^{-\lambda_k t}$, we can write

$$\left\{\begin{aligned} \delta_{kl} &= (\varphi_k\,,\,\widetilde{q}_l)_{L^2(0,\infty;\mathbb{C})} = (R_T^{-1}R_T\varphi_k\,,\,\widetilde{q}_l)_{L^2(0,\infty;\mathbb{C})} \\ &= (R_T\varphi_k\,,\,\left(R_T^{-1}\right)^*\widetilde{q}_l)_{L^2(0,T;\mathbb{C})} = (\varphi_k\,,\,q_l)_{L^2(0,T;\mathbb{C})}, \quad \forall k,l \geq 1, \end{aligned}\right.$$

i.e., $\{q_k\}_{k\geq 1} \subset A(\Lambda, T)$ is a biorthogonal family to $\{\varphi_k\}_{k\geq 1}$ in $L^2(0, T; \mathbb{C})$. This ends the proof of Theorem 4.1. $\square$

Remark 4.3. It is interesting to point out that when $c\,(\Lambda) < \infty$, inequality (4.3) can be equivalently written under the form: For any $\varepsilon > 0$ there exist positive constants $C_{1,\varepsilon}, C_{2,\varepsilon}$ such that

$$C_{2,\varepsilon}e^{(c(\Lambda)-\varepsilon)\Re(\lambda_k)} \leq \|q_k\|_{L^2(0,T;\mathbb{C})} \leq C_{1,\varepsilon}e^{(c(\Lambda)+\varepsilon)\Re(\lambda_k)}, \quad \forall k \geq 1.$$

In this sense, the condensation index of the sequence Λ measures the growth of the L^2-norm of the biorthogonal family q_k with respect to $\Re\,(\lambda_k)$. This inequality and inequality (4.3) will play an important role in the proof of the positive null controllability part of Theorem 2.6.

On the other hand, let us consider $\{q_k\}_{k\geq 1}$ a biorthogonal family to the exponentials $\left\{e^{-\lambda_k t}\right\}_{k\geq 1}$ in $L^2(0, T; \mathbb{C})$. Then,

$$\begin{aligned} 1 &= \int_0^T e^{-\lambda_k t}\overline{q}_k(t)\,dt \leq \|q_k\|_{L^2(0,T;\mathbb{C})}\|e^{-\lambda_k t}\|_{L^2(0,T;\mathbb{C})} \\ &\leq \|q_k\|_{L^2(0,T;\mathbb{C})}\|e^{-\lambda_k t}\|_{L^2(0,\infty;\mathbb{C})} \equiv \frac{1}{\sqrt{2\Re(\lambda_k)}}\|q_k\|_{L^2(0,T;\mathbb{C})}\,. \end{aligned}$$

If we put this inequality together with inequality (4.3), we deduce that, for any $\varepsilon > 0$, there exists a positive constant C_ε such that

$$\frac{1}{|E'(\lambda_k)|} \geq C_\varepsilon \sqrt{2\Re(\lambda_k)} e^{-\varepsilon \Re(\lambda_k)}, \quad \forall k \geq 1. \tag{4.5}$$

We will use this inequality below. □

Remark 4.4. In Theorem 4.1 we have proved that, under assumption (2.2), $A(\Lambda, \infty)$ is a closed proper subspace of $L^2(0, \infty; \mathbb{C})$. In fact, the existence of a biorthogonal family $\{q_k\}_{k\geq 1}$ to the exponentials $\{e^{-\lambda_k t}\}_{k\geq 1}$ in $L^2(0, \infty; \mathbb{C})$ implies that the set $\{e^{-\lambda_k t}\}_{k\geq 1}$ forms a strongly independent set in $L^2(0, \infty; \mathbb{C})$, i.e., each element $e^{-\lambda_k t}$ of this set is outside the closure of the space spanned by the other functions of the set. Thanks to Lemma 4.2, these last results can be easily generalized to the case $T \in (0, \infty)$. □

Our next task will be to prove Lemma 4.2. The proof is technical and needs some preliminary results.

Let us consider $\Lambda = \{\lambda_k\}_{k\geq 1} \subset \mathbb{C}$ a sequence satisfying (2.2). From assumption (2.2) we deduce the existence of $\theta_\delta \in [0, \pi/2)$ such that

$$\Lambda = \{\lambda_k\}_{k\geq 1} \subset S_\delta := \{z = re^{i\theta} \in \mathbb{C} : r > 0, \ |\theta| \leq \theta_\delta\}. \tag{4.6}$$

We begin by recalling a result on the asymptotic behavior of the Blaschke product W defined in (3.18). This result reads as follows:

Proposition 4.5. *Let $\Lambda = \{\lambda_k\}_{k\geq 1} \subset \mathbb{C}$ be a sequence satisfying (2.2). Then, for a fixed θ_0, $0 \leq \theta_0 < \pi/2$, one has:*

1. *There exists an increasing sequence of positive numbers $\{r_n\}_{n\geq 1}$ such that $\lim r_n = \infty$ and*

$$\lim r_n^{-1} \log \left| W\left(r_n e^{i\theta}\right)\right| = 0, \tag{4.7}$$

uniformly in $|\theta| \leq \theta_0$.

2. *Let us consider $\theta \in (-\pi/2, \pi/2)$ such that $\{z = re^{i\theta} \in \mathbb{C} : r > 0\} \cap \Lambda = \emptyset$. Then:*

$$\lim_{r\to\infty} r^{-1} \log \left| W\left(re^{i\theta}\right)\right| = 0.$$

□

For a proof of this result, see [6, Theorem 7.2.3., p. 115].

We will also need a result on the asymptotic behavior of Dirichlet polynomials associated with the sequence $\Lambda = \{\lambda_k\}_{k\geq 1} \subset \mathbb{C}$. Let us set

$$\mathfrak{P} := \left\{ P : P(z) = \sum_{j=1}^{N} a_j e^{-\lambda_j z}, \ \forall z : \Re(z) > 0, \ \text{with } N \geq 1, \ \text{and } a_j \in \mathbb{C} \right\}.$$

The result is the following one:

Lemma 4.6. *Let $\Lambda = \{\lambda_k\}_{k\geq 1} \subset \mathbb{C}$ be a sequence satisfying (2.2). Let us consider $\theta_\delta \in [0, \pi/2)$ such that (4.6) holds. Let us also fix $\theta_0 \in (\theta_\delta, \pi/2)$ and $\varepsilon > 0$, and define the sector*

$$S_{\varepsilon,\theta_0,\tau} = \left\{ z = x + iy : x \geq \varepsilon,\ \frac{|y|}{x} \leq \frac{\cos\theta_0 - \tau}{\sin\theta_0} \right\}$$

with $\tau \in (0, \cos\theta_0)$. Then, there exists a constant $C_\varepsilon > 0$ such that, for any $P \in \mathfrak{P}$, one has

$$|P(z)| \leq C_\varepsilon \|P\|_{L^2(0,\infty;\mathbb{C})} e^{-\frac{1}{4}|\lambda_1|\tau \Re(z)}, \quad \forall z \in S_{\varepsilon,\theta_0,\tau},$$

where $|\lambda_1| = \min_{k\geq 1} |\lambda_k|$. □

We will first give the proof of Lemma 4.2 assuming that Lemma 4.6 has been proved. Then, we will present the proof of this last result.

Proof of Lemma 4.2. Let us consider a sequence $\Lambda = \{\lambda_k\}_{k\geq 1} \subset \mathbb{C}$ satisfying (2.2). Taking into account that $A(\Lambda, T) = \overline{\text{span}\ \{e^{-\lambda_k t} : k \geq 1\}}^{L^2(0,T;\mathbb{C})}$, it suffices to prove the existence of a positive constant C_T for which

$$\|P\|_{L^2(0,\infty;\mathbb{C})} \leq C_T \|P\|_{L^2(0,T;\mathbb{C})}, \quad \forall P \in \mathfrak{P}.$$

We proceed by contradiction. Assume that there exists a sequence $\{P_m\}_{m\geq 1} \subset \mathfrak{P}$ such that

$$\lim \|P_m\|_{L^2(0,T;\mathbb{C})} = 0 \quad \text{and} \quad \|P_m\|_{L^2(0,\infty;\mathbb{C})} = 1 \quad \forall m \geq 1. \tag{4.8}$$

Let $\theta_0 \in (\theta_\delta, \pi/2)$ where $\theta_\delta \in (0, \pi/2)$ is such that (4.6) holds. Let us also fix $\varepsilon > 0$ and $\theta_0 \in (\theta_\delta, \pi/2)$ and $\tau \in (0, \cos\theta_0)$. Using Lemma 4.6 and (4.8), we can conclude that the sequence $\{P_m\}_{m\geq 1}$ is uniformly bounded on the domain $S_{\varepsilon,\theta_0,\tau}$. Therefore, it is a normal family of holomorphic functions on $S_{\varepsilon,\theta_0,\tau}$ and there exists a subsequence, still denoted by $\{P_m\}_{m\geq 1}$, and a holomorphic function P on $S_{\varepsilon,\theta_0,\tau}$ such that $P_m \to P$ uniformly on the compact sets of $S_{\varepsilon,\theta_0,\tau}$. Furthermore, from Lebesgue's theorem, $P_m \to P$ in $L^2(\eta, \infty; \mathbb{C})$ for any $\eta > \varepsilon$. Assumption (4.8) implies that $P \equiv 0$ on the interval (η, T) for any $\eta : 0 < \varepsilon < \eta < T$. Since P is holomorphic on $S_{\varepsilon,\theta_0,\tau}$, we get $P \equiv 0$ on (ε, ∞). Whence $\lim \|P_m\|_{L^2(T,\infty;\mathbb{C})} = 0$ and since, by our assumption, $\lim \|P_m\|_{L^2(0,T;\mathbb{C})} = 0$ it follows that

$$\lim \|P_m\|_{L^2(0,\infty;\mathbb{C})} = 0.$$

This contradicts (4.8) and provides the proof of Lemma 4.2. □

Proof of Lemma 4.6. Given the sequence $\Lambda = \{\lambda_k\}_{k\geq 1} \subset \mathbb{C}$ satisfying (2.2), let us fix $\theta_\delta \in [0, \pi/2)$ such that (4.6) holds, $\theta_0 \in (\theta_\delta, \pi/2)$ and $\varepsilon > 0$. We can apply Proposition 4.5 and deduce the existence of a sequence $\{r_n\}_{n\geq 1} \subset \mathbb{R}_+$ satisfying $\lim r_n = \infty$ and (4.7).

Observe that $W(\lambda_k) = 0$, for any $k \geq 1$, and thus, $\{|\lambda_n|\}_{n\geq 1} \cap \{r_n\}_{n\geq 1} = \emptyset$. So, we can assume that the sequence $\{r_n\}_{n\geq 1}$, or a subsequence, is increasing and such that for each $n \geq 1$, the set

$$G_n := \{z = re^{i\theta} : r_n < |z| < r_{n+1},\ |\theta| < \theta_0\}$$

contains at least an element of the sequence Λ. We can also assume that $r_1 = \frac{1}{2}|\lambda_1|$, where $|\lambda_1| = \min_{k\geq 1} |\lambda_k|$, and

$$\Lambda \subset \bigcup_{k\geq 1} G_k\,.$$

Let P be any Dirichlet polynomial:

$$P(z) = \sum_{j=1}^{N} c_j e^{-\lambda_j z}, \quad \forall z \in \mathbb{C}_+ ,$$

where $c_j \in \mathbb{C}$, for $j \geq 1$. Then there exists $m = m(N, \Delta) \geq 1$ such that $\{\lambda_j\}_{1\leq j\leq N} \subset \cup_{k=1}^{m} G_k$ and $P \in A(\Lambda, \infty)$ can be written in the form:

$$P(z) = \sum_{k=1}^{m} \sum_{\lambda_n \in G_k} c_n e^{-\lambda_n z} = \sum_{k=1}^{m} g_k(z), \quad \forall z \in \mathbb{C}_+ . \tag{4.9}$$

Recall that the biorthogonal family $\{\widetilde{q}_k\}_{k\geq 1}$ to $\left\{e^{-\lambda_k t}\right\}_{k\geq 1}$ in $L^2(0,\infty;\mathbb{C})$ was constructed in Theorem 4.1 from the formula (4.4). So that the coefficients c_n of P are given by

$$c_n = \int_0^{\infty} P(t)\overline{\widetilde{q}}_n(t)\, dt.$$

Coming back to the expression (4.9) of P, we deduce

$$g_k(z) = \int_0^{\infty} P(t) \sum_{\lambda_n \in G_k} \overline{\widetilde{q}}_n(t) e^{-\lambda_n z}\, dt, \quad \forall z \in \mathbb{C}_+ ,$$

and by Schwartz inequality:

$$|g_k(z)|^2 \leq \|P\|^2_{L^2(0,\infty;\mathbb{C})} \int_0^{\infty} \left| \sum_{\lambda_n \in G_k} \overline{\widetilde{q}}_n(t) e^{-\lambda_n z} \right|^2 dt := \|P\|^2_{L^2(0,\infty;\mathbb{C})} \int_0^{\infty} |\mathcal{G}_k(t,z)|^2 \, dt. \tag{4.10}$$

Using again (4.4), we can calculate the Laplace transform of $\mathcal{G}_k$ which is given by:

$$\int_0^{\infty} e^{-\lambda t} \mathcal{G}_k(t,z)\, dt = \sum_{\lambda_n \in G_k} J_n(\lambda) e^{-\lambda_n z},$$

Applying now the Parseval equality, we also get

$$\frac{1}{2\pi} \int_0^{\infty} |\mathcal{G}_k(t,z)|^2 \, dt = \int_{-\infty}^{+\infty} \left| \sum_{\lambda_n \in G_k} J_n(i\tau) e^{-\lambda_n z} \right|^2 d\tau.$$

Whence, inequality (4.10) writes:

$$|g_k(z)|^2 \leq 2\pi \|P\|^2_{L^2(0,\infty;\mathbb{C})} \int_{-\infty}^{+\infty} \left| \sum_{\lambda_n \in G_k} J_n(i\tau) e^{-\lambda_n z} \right|^2 d\tau. \tag{4.11}$$

Let Γ_k be the boundary of G_k. Since each λ_n is a simple zero of J, from the residue theorem, we obtain:

$$\sum_{\lambda_n \in G_k} J_n(i\tau) e^{-\lambda_n z} = \frac{J(i\tau)}{2i\pi} \int_{\Gamma_k} \frac{e^{-\xi z}}{J(\xi)(i\tau - \xi)}\, d\xi.$$

Then:

$$\int_{-\infty}^{+\infty} \left| \sum_{\lambda_n \in G_k} J_n(i\tau) e^{-\lambda_n z} \right|^2 d\tau \leq \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} |J(i\tau)|^2 \left(\int_{\Gamma_k} \left| \frac{e^{-\xi z}}{J(\xi)(i\tau - \xi)} \right| |d\xi| \right)^2 d\tau$$

$$\leq \frac{1}{4\rho^2\pi^2} \left(\int_{\Gamma_k} \left| \frac{e^{-\xi z}}{J(\xi)} \right| |d\xi| \right)^2 \int_{-\infty}^{+\infty} |J(i\tau)|^2 \, d\tau$$

$$= \frac{\|J\|_{H^2}^2}{4\rho^2\pi^2} \left(\int_{\Gamma_k} \left| \frac{e^{-\xi z}}{J(\xi)} \right| |d\xi| \right)^2,$$

where $|i\tau - \xi| \geq \rho = \min_{k\geq 1} \operatorname{dist}(\Gamma_k, i\mathbb{R}) > 0$ for any $\xi \in \Gamma_k$ and any $k \geq 1$. Inserting this last inequality in (4.11), we deduce the estimate:

$$|g_k(z)| \leq \frac{\|J\|_{H^2}}{\rho\sqrt{2\pi}} \|P\|_{L^2(0,\infty;\mathbb{C})} \int_{\Gamma_k} \left| \frac{e^{-\xi z}}{J(\xi)} \right| |d\xi|, \quad \forall z \in \mathbb{C}_+ .$$

Going back to the expression (4.9) of $P(z)$, we finally obtain the estimate

$$|P(z)| \leq \frac{\|J\|_{H^2}}{\rho\sqrt{2\pi}} \|P\|_{L^2(0,\infty;\mathbb{C})} \sum_{k=1}^{m} \int_{\Gamma_k} \left| \frac{e^{-\xi z}}{J(\xi)} \right| |d\xi|, \quad \forall z \in \mathbb{C}_+ . \tag{4.12}$$

Let us now work with the previous integral on Γ_k. At this level we will use the properties of the set G_k and, in particular, the properties of the sequence $\{r_n\}_{n\geq 1}$.

The boundary Γ_k can be divided into four subsets, namely, $\Gamma_k = \Gamma_k^{\pm\theta_0} \cup \Gamma_k^{r_k} \cup \Gamma_k^{r_{k+1}}$, where

$$\Gamma_k^{\pm\theta_0} = \{z = re^{i\theta} : r_k < |z| < r_{k+1}, \ |\theta| = \pm\theta_0\}, \quad \Gamma_k^{r_k} = \{z = |z|e^{i\theta} : |z| = r_k, \ |\theta| \leq \theta_0\}.$$

The assumption (2.2) and the choice of $\theta_0 \in (\theta_\delta, \pi/2)$ (where θ_δ is such that (4.6) holds) implies that the lines $\{z = re^{\pm i\theta_0} : r > 0\}$ do not intersect the sequence Λ. So, the second point of Proposition 4.5 can be applied for $\theta = \pm\theta_0$. On the other hand, the sequence $\{r_n\}_{n\geq 1}$ satisfies (4.7). From this two properties, we obtain that for any $\eta > 0$ there exists $k_0 = k_0(\eta)$ such that

$$e^{-\eta|\xi|} \leq |W(\xi)| \leq e^{\eta|\xi|}, \quad \forall \xi \in \Gamma_k, \quad \forall k > k_0 . \tag{4.13}$$

If $z = x + iy \in S_{\varepsilon,\theta_0,\tau}$ and $\xi = re^{i\theta} \in \Gamma_k$, we can write

$$x\cos\theta - y\sin\theta \geq x\cos\theta - |y||\sin\theta| \geq x\cos\theta - |y|\sin\theta_0 \geq \tau x,$$

and

$$\left| e^{-\xi z} \right| = e^{-r(x\cos\theta - y\sin\theta)} \leq e^{-r\tau x} = e^{-|\xi|\tau\Re(z)}, \quad \forall z \in S_{\varepsilon,\theta_0,\tau}, \quad \forall \xi \in \Gamma_k, \quad \forall k \geq 1. \tag{4.14}$$

In the sequel, C will denote a generic positive constant; sometimes, we will lay emphasis on the dependence of C on η (resp., ε), by writing C_η (resp., C_ε).

As said before, the properties of θ_0 and the sequence $\{r_n\}_{n\geq 1}$ ensures that $\Gamma_k \cap \Lambda = \emptyset$ for all $k \geq 1$. Thus, $W(\xi) \neq 0$ for any $\xi \in \Gamma_k$ and we deduce that there exists a positive constant C_η such that

$$|W(\xi)| \geq C_\eta, \quad \forall \xi \in \Gamma_k, \quad \forall k : 1 \leq k \leq k_0 .$$

Taking into account this bound, the expression of the function J, see (4.1), and (4.14), we get

$$\int_{\Gamma_k}\left|\frac{e^{-\xi z}}{J(\xi)}\right|\,|d\xi|\le 2C_\eta\int_{\Gamma_k}\left|e^{-\xi z}\right|\left(1+|\xi|^2\right)\,|d\xi|\le C_\eta\int_{\Gamma_k}e^{-|\xi|\tau\Re(z)}\,|d\xi|\le C_\eta e^{-r_1\tau\Re(z)},$$

for any $z\in S_{\varepsilon,\theta_0,\tau}$ and any k, with $1\le k\le k_0$. Recall that we took $r_1=\frac{1}{2}|\lambda_1|\ge\frac{1}{4}|\lambda_1|$. In conclusion,

$$\int_{\Gamma_k}\left|\frac{e^{-\xi z}}{J(\xi)}\right|\,|d\xi|\le C_\eta e^{-\frac{1}{4}|\lambda_1|\tau\Re(z)},\quad\forall z\in S_{\varepsilon,\theta_0,\tau},\quad\forall k:1\le k\le k_0. \tag{4.15}$$

Let us now consider $k>k_0$. Taking into account (4.13) and (4.14), for $z\in S_{\varepsilon,\theta_0,\tau}$ we can also bound

$$\begin{aligned}\int_{\Gamma_k}\left|\frac{e^{-\xi z}}{J(\xi)}\right|\,|d\xi|&\le 2\int_{\Gamma_k}e^{-|\xi|\left(\frac{1}{2}\tau\Re(z)-\eta\right)}e^{-\frac{1}{2}|\xi|\tau\Re(z)}\left(1+|\xi|^2\right)\,|d\xi|\\&\le 2e^{-\frac{1}{4}|\lambda_1|\tau\Re(z)}\int_{\Gamma_k}e^{-|\xi|\left(\frac{1}{2}\tau\varepsilon-\eta\right)}\left(1+|\xi|^2\right)\,|d\xi|,\quad\forall k>k_0\end{aligned}$$

We can now determine the parameter η. To be precise, let us take $\eta=\frac{1}{4}\tau\varepsilon$. With this value, the previous inequality can be written as

$$\int_{\Gamma_k}\left|\frac{e^{-\xi z}}{J(\xi)}\right|\,|d\xi|\le 2e^{-\frac{1}{4}|\lambda_1|\tau\Re(z)}\int_{\Gamma_k}e^{-\frac{1}{4}|\xi|\tau\varepsilon}\left(1+|\xi|^2\right)\,|d\xi|,\quad\forall z\in S_{\varepsilon,\theta_0,\tau},\quad\forall k>k_0\,.$$

Observe that $\Gamma_k=\Gamma_k^{\pm\theta_0}\cup\Gamma_k^{r_k}\cup\Gamma_k^{r_{k+1}}$. Thus,

$$\int_{\Gamma_k^{\pm\theta_0}}e^{-\frac{1}{4}|\xi|\tau\varepsilon}\left(1+|\xi|^2\right)\,|d\xi|=\int_{r_k}^{r_{k+1}}e^{-\frac{1}{4}\tau\varepsilon r}\left(1+r^2\right)\,dr.$$

and

$$\int_{\Gamma_k^{r_k}}e^{-\frac{1}{4}|\xi|\tau\varepsilon}\left(1+|\xi|^2\right)\,|d\xi|=e^{-\frac{1}{4}r_k\tau\varepsilon}\left(1+r_k^2\right)\int_{r_ke^{-i\theta_0}}^{r_ke^{i\theta_0}}|d\xi|\le Cr_k\left(1+r_k^2\right)e^{-\frac{1}{4}r_k\tau\varepsilon}.$$

Summarizing, we have obtain

$$\begin{aligned}\int_{\Gamma_k}\left|\frac{e^{-\xi z}}{J(\xi)}\right|\,|d\xi|\le Ce^{-\frac{1}{4}|\lambda_1|\tau\Re(z)}\Bigg(&r_k\left(1+r_k^2\right)e^{-\frac{1}{4}r_k\tau\varepsilon}+r_{k+1}\left(1+r_{k+1}^2\right)e^{-\frac{1}{4}r_{k+1}\tau\varepsilon}\\&+\int_{r_k}^{r_{k+1}}e^{-\frac{1}{4}\tau\varepsilon r}\left(1+r^2\right)\,dr\Bigg),\quad\forall z\in S_{\varepsilon,\theta_0,\tau},\quad\forall k>k_0\,.\end{aligned}$$

Putting (4.15) and this last inequality in (4.12), we can write:

$$\begin{aligned}|P(z)|\le C_\varepsilon e^{-\frac{1}{4}|\lambda_1|\tau\Re(z)}\|J\|_{H^2}\|P\|_{L^2(0,\infty;\mathbb{C})}\Bigg(&1+\sum_{k=1}^{m+1}r_k\left(1+r_k^2\right)e^{-\frac{1}{4}r_k\tau\varepsilon}\\&+\int_{r_1}^{r_{m+1}}e^{-\frac{1}{4}\tau\varepsilon r}\left(1+r^2\right)\,dr\Bigg),\quad\forall z\in S_{\varepsilon,\theta_0,\tau}\,.\end{aligned}$$

Finally, recall that the sequence $\{r_n\}_{n\geq 1} \subset (0,\infty)$ is increasing and satisfies $\lim r_n = \infty$. Then, the function $\beta(r) = e^{-\frac{1}{4}\tau\varepsilon r}\left(1+r^2\right)$, with $r \in \mathbb{R}_+$, satisfies $\beta \in L^1(0,\infty)$ and the series

$$\sum_{k\geq 1} r_k\left(1+r_k^2\right)e^{-\frac{1}{4}r_k\tau\varepsilon}$$

is convergent. We can thus conclude that for a new constant $C_\varepsilon > 0$ one has

$$|P(z)| \leq C_\varepsilon e^{-\frac{1}{4}|\lambda_1|\tau\Re(z)}\|P\|_{L^2(0,\infty;\mathbb{C})}, \quad \forall z \in S_{\varepsilon,\theta_0,\tau}\,.$$

This ends the proof. □

Remark 4.7. Lemmata 4.2 and 4.6 have been first proved by L. Schwartz (see [28]) in the case of a real increasing sequence $\{\lambda_n\}_{n\geq 1}$ of positive numbers such that

$$\sum_{n\geq 1}\frac{1}{\lambda_n} < \infty.$$

For complex sequences, similar results have been proved by S. Hansen in [15] and by E. Fernández-Cara and al. in [12]. In these two last articles, the complex sequences had zero condensation index and this fact is strongly used in the respective proofs of these authors. □

Let us end this section by giving some properties of the minimal time T_0 (see (2.13)) which, in particular, relate this number to the condensation index of the corresponding complex sequence Λ. We will use Theorem 4.1 in a fundamental way. One has:

Theorem 4.8. *Let us assume the hypotheses of Theorem 2.6 and let T_0 be the number given by (2.13). Then, $T_0 \in [\max\{T_1, c(\Lambda)\}, T_1 + c(\Lambda)] \subseteq [0,\infty]$ where $c(\Lambda)$ is given in (3.2) and*

$$T_1 := \limsup \frac{\log\frac{1}{|b_k|}}{\Re(\lambda_k)}.$$

In particular, if $T_1 = 0$ (resp., $c(\Lambda) = 0$) then, $T_0 = c(\Lambda)$ (resp., $T_0 = T_1$).

Proof. Since $b_k = \mathcal{B}^*\psi_k$, with $\mathcal{B}^* \in \mathbb{X}_{-1}$, and satisfies (2.10), we deduce that there exists a constant $\sigma > 0$ such that

$$0 < |b_k| \leq \sigma|\lambda_k| \leq \frac{\sigma}{\delta}\Re(\lambda_k), \quad \forall k \geq 1.$$

Thus,

$$\frac{\log\frac{1}{|b_k|}}{\Re(\lambda_k)} \geq \frac{\log\left(\frac{\delta}{\sigma}\frac{1}{\Re(\lambda_k)}\right)}{\Re(\lambda_k)}, \quad \forall k \geq 1.$$

Using the assumption (2.2) on the sequence Λ, it follows that $T_1 \geq 0$ and the property: for any $\varepsilon > 0$, there exists $k_0(\varepsilon) \geq 1$ such that

$$\frac{\log\frac{1}{|b_k|}}{\Re(\lambda_k)} \geq -\varepsilon, \quad \forall k \geq k_0(\varepsilon).$$

On the other hand, let us fix ε, a positive (and arbitrary) constant. From inequality (4.5) we get,

$$\frac{\log\frac{1}{|E'(\lambda_k)|}}{\Re(\lambda_k)} \geq \frac{\log\left(C_\varepsilon\sqrt{2\Re(\lambda_k)}e^{-\varepsilon\Re(\lambda_k)}\right)}{\Re(\lambda_k)}, \quad \forall k \geq 1.$$

As above, we deduce again that $c(\Lambda) \geq 0$ and the existence of $k_1(\varepsilon) \geq 1$ such that

$$\frac{\log \frac{1}{|E'(\lambda_k)|}}{\Re(\lambda_k)} \geq -\varepsilon, \quad \forall k \geq k_1(\varepsilon).$$

Now fix a subsequence $\{k_n\}_{n\geq 1}$ of integers such that

$$\lim_{n\to\infty} \frac{\log \frac{1}{|b_{k_n}|}}{\Re(\lambda_{k_n})} = T_1.$$

Then, for n sufficiently large:

$$\frac{\log \frac{1}{|b_{k_n}|}}{\Re(\lambda_{k_n})} + \frac{\log \frac{1}{|E'(\lambda_{k_n})|}}{\Re(\lambda_{k_n})} \geq T_1 - 2\varepsilon,$$

and this implies that $T_0 \geq T_1 - 2\varepsilon$. Since $\varepsilon > 0$ can be chosen arbitrarily small, we conclude that $T_0 \geq T_1$.

The inequality $T_0 \geq c(\Lambda)$ can be obtained in the same way. This ends the proof of the result. □

Remark 4.9. As a consequence of the proof of Theorem 4.8 we deduce that, under the assumption of Theorem 2.6, $T_1, c(\Lambda) \in [0,\infty]$. In fact, in Subsections 6.1 and 6.2 we will see that T_1 and $c(\Lambda)$ can take any value in the interval $[0,\infty]$. □

5 Proof of the main result

We will devote this section to proving the main result of this paper, Theorem 2.6. Let us then consider System (2.3) where $\mathcal{A}$ is given by (2.4) ($\Lambda = \{\lambda_k\}_{k\geq 1}$ is a sequence satisfying (2.2)) and $\mathcal{B} \in \mathcal{L}(\mathbb{C}, \mathbb{X}_{-1})$ is an admissible control operator for the semigroup $\left\{e^{t\mathcal{A}}\right\}_{t>0}$ generated by $\mathcal{A}$. In addition, let us assume that the coefficients $b_k = \mathcal{B}^*\psi_k$ fulfills condition (2.10). We will divide the proof into two parts, the first one will contain the positive null controllability result for System (2.3) whereas in the second part we will show the negative null controllability result for this system.

5.1 Proof of the null controllability part of Theorem 2.6

In order to prove this first part of Theorem 2.6 we will assume that $T > T_0$, with T_0 given by (2.13). Observe that in this case $T_0 \in [0,\infty)$. We want to prove that for any $y_0 \in \mathbb{X}$ there exists $u \in L^2(0,T;\mathbb{C})$ such that the corresponding solution $y \in C^0([0,T];\mathbb{X})$ to (2.3) satisfies $y(T) = 0$ in $\mathbb{X}$. From the expression (2.7), this amounts to

$$\int_0^T e^{(T-s)\mathcal{A}}\mathcal{B}u(s)\,ds = -e^{T\mathcal{A}}y_0 \quad \text{in} \quad \mathbb{X}.$$

Since the sequence of eigenvalues of $-\mathcal{A}$, $\{\lambda_k\}_{k\geq 1}$, are pairwise distinct and the set $\{\psi_k\}_{k\geq 1}$ is a Riesz basis of $\mathbb{X}$, this last problem is equivalent to:

$$\int_0^T \left(u, \mathcal{B}^* e^{(T-t)\mathcal{A}^*}\psi_k\right) dt = -\left(y_0, e^{T\mathcal{A}^*}\psi_k\right), \quad \forall k \geq 1.$$

Now, using the expression of $\mathcal{A}^*$ (see (2.5)), we readily deduce that the null controllability problem for System (2.3) reduces to the following moment problem: Find $u \in L^2(0,T;\mathbb{C})$ such that

$$\bar{b}_k \int_0^T e^{-\lambda_k t} u(T-t)\, dt = -e^{-\lambda_k T} \left(y_0, \psi_k\right), \quad \forall k \geq 1, \tag{5.1}$$

with b_k given by (2.10). Clearly, the assumption (2.10) on the coefficient b_k is a necessary condition for having a solution of the previous moment problem for arbitrary initial data $y_0 \in \mathbb{X}$.

As said before, we will solve the moment problem (5.1) using a biorthogonal family in $L^2(0,T;\mathbb{C})$ to the complex exponentials $\left\{e^{-\lambda_k t}\right\}_{k\geq 1}$.

Taking into account Theorem 4.1 in the case $T \in (0,\infty)$, we seek a solution $v(t) = u(T-t)$ to (5.1) under the form:

$$v(t) = \sum_{k\geq 1} v_k \overline{q}_k(t),$$

for some unknown coefficients $v_k \in \mathbb{C}$ ($k \geq 1$). This leads to the formal solution:

$$v_k = -\frac{e^{-\lambda_k T}}{\bar{b}_k} \left(y_0, \psi_k\right), \quad \forall k \geq 1,$$

this is to say,

$$u(t) = v(T-t) = -\sum_{k\geq 1} \frac{e^{-\lambda_k T}}{\bar{b}_k} \left(y_0, \psi_k\right) \overline{q}_k(T-t).$$

Observe that the null controllability problem for System (2.3) is solved if we could prove that $u \in L^2\left(0,T;\mathbb{C}\right)$, i.e., if we prove that the previous series converges in $L^2\left(0,T;\mathbb{C}\right)$. At this point we will use Theorem 4.1. Given $\varepsilon > 0$, we can apply Theorem 4.1 and obtain, for any $k \geq 1$,

$$\begin{aligned}\left\| \frac{e^{-\lambda_k T}}{\bar{b}_k} \left(y_0, \psi_k\right) \overline{q}_k \right\|^2_{L^2(0,T;\mathbb{C})} &= \frac{e^{-2\Re(\lambda_k)T}}{\left|b_k\right|^2} \left|\left(y_0, \psi_k\right)\right|^2 \left\|q_k\right\|^2_{L^2(0,T;\mathbb{C})} \\ &\leq C_\varepsilon e^{-2\Re(\lambda_k)\left[T - \frac{\log(1/|b_k|)+\log(1/|E'(\lambda_k)|)}{\Re(\lambda_k)} - \varepsilon\right]} \left|\left(y_0, \psi_k\right)\right|^2,\end{aligned}$$

where C_ε is a positive constant. It follows that if $T > T_0$, with T_0 given by (2.13), and if we choose $\varepsilon \in (0, (T-T_0)/2)$, then the previous inequality leads to:

$$\left\| \frac{e^{-\lambda_k T}}{\bar{b}_k} \left(y_0, \psi_k\right) \overline{q}_k \right\|^2_{L^2(0,T;\mathbb{C})} \leq C_\varepsilon e^{-2\Re(\lambda_k)(T-T_0-2\varepsilon)} \left|\left(y_0, \psi_k\right)\right|^2, \quad \forall k \geq k_\varepsilon,$$

with $k_\varepsilon \geq 1$. As a consequence, we deduce that u is an absolutely convergent series in $L^2(0,T;\mathbb{C})$ and thus $u \in L^2(0,T;\mathbb{C})$ with

$$\|u\|_{L^2(0,T;\mathbb{C})} \leq \left(\sum_{k\geq 1} \left\| \frac{e^{-\lambda_k T}}{\bar{b}_k} \left(y_0, \psi_k\right) \overline{q}_k \right\|^2_{L^2(0,T;\mathbb{C})} \right)^{1/2} \left\|y_0\right\|_0,$$

(the norm $\|\cdot\|_0$ is defined in (2.1)). In conclusion, we have proved that System (2.3) is null controllable in $\mathbb{X}$ at any time $T > T_0$. This concludes the first part of the proof of Theorem 2.6. $\square$

5.2 Proof of the second part of Theorem 2.6

In this section we are going to prove the second part of Theorem 2.6. In order to achieve this aim, let us assume that $T \in (0, T_0)$ with T_0 given by (2.13) (and implicitly, $T_0 > 0$). The objective is to prove that System (2.3) is not null controllable in $\mathbb{X}$ at time T. Before, let us see a general property for condensation groupings associated with complex sequences. It reads as follows:

Proposition 5.1. *Let $\Lambda = \{\lambda_k\}_{k\geq 1} \subset \mathbb{C}$ and $\Delta = \{G_k\}_{k\geq 1}$ be, resp., a normally ordered sequence satisfying (2.2) and a condensation grouping of this sequence Λ. Then,*

$$\lim \int_0^\infty \left| \sum_{\lambda_n \in G_k} \frac{p_k!}{P'_{G_k}(\lambda_n)} e^{-\lambda_n t} \right|^2 dt = 0, \tag{5.2}$$

where $p_k + 1$ is the cardinal of the set $G_k \cap \Lambda$ and P_{G_k} is given by (3.3).

Proof. The proof of this result requires the following formula due to Jensen [20]:

Lemma 5.2. *Let $A = \{a_j\}_{0\leq j\leq q} \subset \mathbb{C}$ be a set of distinct points and let us fix f an analytic function in a convex domain $\Omega \subset \mathbb{C}$ such that $A \subset \Omega$. Then, there exist $\theta \in [-1, 1]$ and $\xi \in \operatorname{Conv}(A)$, the convex hull of A, such that*

$$\sum_{j=0}^{q} \frac{f(a_j)}{P'_A(a_j)} = \frac{\theta}{q!} \frac{d^q f}{dz^q}(\xi), \tag{5.3}$$

where P_A is given by (3.3). □

Going back to the proof of the lemma, we fix $t \in (0, \infty)$ and we apply formula (5.3) with $f(z) = e^{-tz}$, $q = p_k$ and $A = G_k \cap \Lambda$. Hence,

$$S_k(t) := p_k! \sum_{\lambda_n \in G_k} \frac{e^{-\lambda_n t}}{P'_{G_k}(\lambda_n)} = \theta_k \, t^{p_k} e^{-t\xi_k}$$

with $|\theta_k| \leq 1$ and

$$\xi_k = \sum_{\lambda_n \in G_k} \alpha_n \lambda_n, \quad \text{with } \alpha_n \geq 0 \text{ and } \sum_{\lambda_n \in G_k} \alpha_n = 1.$$

Now, if λ_{l_k} is the first element of $G_k \cap \Lambda$, we have

$$|S_k(t)| \leq t^{p_k} e^{-\Re(\xi_k)t} \leq e^{tp_k} e^{-\delta|\lambda_{l_k}|t} = e^{-t|\lambda_{l_k}|[\delta - p_k/|\lambda_{l_k}|]} := g_k(t).$$

In this inequality we have used assumption (2.2) and the inequality $\log t \leq t$, valid for any $t > 0$. Let us recall that $p_k \geq 0$ and it satisfies (see Definition 3.4 (iii))

$$\lim \frac{p_k}{|\lambda_{l_k}|} = 0.$$

Therefore, there exists a positive constant C_δ such that $g_k(t) \leq C_\delta e^{-t|\lambda_{l_k}|\delta/2}$ for any $t \in (0, \infty)$ and any $k \geq 1$. In particular, it follows that

$$\begin{cases} \lim g_k(t) = 0, \quad \forall t \in (0, +\infty), \\ g_k(t) \leq C_\delta e^{-t|\lambda_1|\delta/2}, \quad \forall k \geq 1, \quad \forall t \in (0, \infty). \end{cases}$$

Thus, Lebesgue's dominated convergence theorem proves (5.2). This ends the proof. □

Let us turn to the proof of the second part of Theorem 2.6. Using Corollary 2.4, the null controllability property for System (2.3) in $\mathbb{X}$ at time T amounts to prove that inequality (2.11) is false for any positive constant $C_T > 0$.

Let us argue by contradiction and assume that there exists a positive constant C_T for which the observability inequality (2.11) holds for any complex sequence $\{a_k\}_{k\geq 1} \subset \ell^2(\mathbb{C})$. As said before, condition (2.10) on the coeficcients b_k is necessary for the null controllability of System (2.3) at time $T > 0$. Therefore, we will assume it.

Let us fix $q \in (0, \infty)$ and a condensation grouping $\Delta = \{G_k\}_{k\geq 1}$ of Λ which satisfies Theorem 3.7. Observe that this condensation grouping also satisfies Theorem 3.8 and the identity (3.5). Let us fix $k \geq 1$ and set

$$a_n^{(k)} = \begin{cases} \dfrac{p_k!}{\overline{b}_n P'_{G_k}(\lambda_n)}, & \text{if } \lambda_n \in G_k. \\ 0 & \text{otherwise,} \end{cases} \tag{5.4}$$

where $p_k + 1$ is the cardinal of G_k and the function P_{G_k} is given by (3.3). Clearly, the (finite) sequence $\{a_n^{(k)}\}_{n\geq 1}$ lies in $\ell^2(\mathbb{C})$ and satisfies the observability inequality (2.11). This observability inequality for $a_n^{(k)}$ can be written:

$$\sigma_k^{(1)} := \sum_{\lambda_n \in G_k} \left| \frac{p_k!}{\overline{b}_n P'_{G_k}(\lambda_n)} e^{-\lambda_n T} \right|^2 \leq C_T \int_0^T \left| \sum_{\lambda_n \in G_k} \frac{p_k!}{P'_{G_k}(\lambda_n)} e^{-\lambda_n t} \right|^2 dt := \sigma_k^{(2)}, \tag{5.5}$$

and this inequality is valid for any $k \geq 1$.

On the one hand, we can apply Proposition 5.1 and from (5.2) we get $\lim \sigma_k^{(2)} = 0$. Our next task will be to prove that $\limsup \sigma_k^{(1)} = \infty$. This clearly provides a contradiction to inequality (5.5) and the proof of the result.

Recall that $T \in (0, T_0)$ and T_0 is given by (2.13). Thus, there exists a subsequence $\{n_k\}_{k\geq 1}$ of positive integers such that

$$T_0 = \lim \left(\frac{\log 1/\left|b_{n_k}\right|}{\Re\left(\lambda_{n_k}\right)} + \frac{\log 1/\left|E'\left(\lambda_{n_k}\right)\right|}{\Re\left(\lambda_{n_k}\right)} \right).$$

At this level we can apply Theorem 3.8 for the subsequence $\{\lambda_{n_k}\}_{k\geq 1} \subseteq \Lambda$. So, if $\{D_k\}_{k\geq 1} \subseteq \Delta$ is a subsequence of sets satisfying $\lambda_{n_k} \in D_k$, for any k, and $q_k + 1$ is the cardinal of the set $D_k \cap \Lambda$, then the identity (3.5) implies

$$T_0 = \lim \frac{1}{\Re(\lambda_{n_k})} \left(\log \left| \frac{1}{b_{n_k}} \right| + \log \left| \frac{q_k!}{P'_{D_k}(\lambda_{n_k})} \right| \right). \tag{5.6}$$

To end the proof, let us show that if $T < T_0$, then $\lim \sigma_{n_k}^{(1)} = \infty$. Indeed,

$$\begin{cases} \sigma_{n_k}^{(1)} = \displaystyle\sum_{\lambda_n \in D_k} \left| \frac{q_k!}{\overline{b}_n P'_{D_k}(\lambda_n)} e^{-\lambda_n T} \right|^2 \geq \left| \frac{q_k!}{\overline{b}_{n_k} P'_{D_k}(\lambda_{n_k})} e^{-\lambda_{n_k} T} \right|^2 \\ \qquad = e^{2\Re(\lambda_{n_k})\left[\frac{1}{\Re(\lambda_{n_k})}\left(\log\left|\frac{1}{b_{n_k}}\right| + \log\left|\frac{q_k!}{P'_{D_k}(\lambda_{n_k})}\right|\right) - T\right]}. \end{cases}$$

This last inequality together with the expression (5.6) of T_0 show $\lim \sigma_{n_k}^{(1)} = \infty$. This contradicts (5.5) and Theorem 2.6 is proved. $\square$

6 Application to some parabolic problem

We will devote this section to presenting some application of Theorem 2.6 to some scalar and non-scalar parabolic problems in the one-dimensional case. First, we will obtain some results for the one-dimensional heat equation with distributed controls. In some particular situations we will obtain controllability results proved in [9]. On the other hand, we will apply Theorem 2.6 to a non-scalar parabolic problem (with boundary and distributed controls) obtaining in this case results which are completely new.

6.1 A distributed controllability problem for the heat equation

Let us consider the one-dimensional heat equation

$$\begin{cases} \partial_t y - \partial_{xx} y = f(x)v(t) & \text{in } Q = (0,\pi)\times(0,T), \\ y(0,\cdot) = 0, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases} \tag{6.1}$$

where $T > 0$, $f \in H^{-1}(0,\pi)$ and $y_0 \in L^2(0,\pi)$ are given and $v \in L^2(0,T)$ is a control to be determined. We are interested in studying the null controllability properties of System (6.1).

The controllability properties of System (6.1) have been intensely studied in the last years by several authors. Among other authors, let us underline H. O. Fattorini and D. L. Russell, who in 1971 and 1974 gave the first results on null controllability for the one-dimensional heat equation (see [10, 11]), and G. Lebeau and L. Robbiano, [24], and A. Fursikov and O. Imanuvilov, [13], who in 1995-1996 solved independently the N-dimensional null controllability problem for parabolic equations (with boundary or distributed controls).

The results we present in this section were obtained by S. Dolecki in [9] in the particular case $f(\cdot) = \delta_{x_0}$, with $x_0 \in (0,\pi)$. In this reference, the author proved the existence of an optimal time $T_0 \in [0,\infty]$, depending on x_0, which provides the null controllability result for System (6.1). The details will be given below.

It is well-known that System (6.1) is well-posed. To be precise, one has:

Proposition 6.1. *There is a positive constant C such that for every $y_0 \in L^2(0,\pi)$, $f \in H^{-1}(0,\pi)$ and $v \in L^2(0,T)$, System (6.1) admits a unique solution $y \in L^2(0,T;H_0^1(0,\pi)) \cap C^0([0,T];L^2(0,\pi))$ which depends continuously on the data:*

$$\|y\|_{L^2(0,T;H_0^1(0,\pi))} + \|y\|_{C^0([0,T];L^2(0,\pi))} \le C\left(\|y_0\|_{L^2(0,\pi)} + \|f\|_{H^{-1}(0,\pi)}\|v\|_{L^2(0,T)}\right).$$

In order to obtain the null controllability resut for System (6.1), let us write the problem under the abstract form (2.3). The objective will be to apply Theorem 2.6.

Let us take $\mathbb{X} = L^2(0,\pi)$ and consider the self-adjoint operator

$$A_0 := -\frac{d^2}{dx^2} : \mathbb{X} \longrightarrow \mathbb{X}$$

with domain $D(A_0) = H^2(0,\pi) \cap H_0^1(0,\pi) \subset \mathbb{X}$. Let us also consider the eigenvalues μ_k and the eigenvectors Φ_k, $k \ge 1$, of the Dirichlet laplacian in $(0,\pi)$, i.e., of A_0 (see (1.3)). Observe that the sequence $\Lambda = \{\mu_k\}_{k\ge1}$ satisfies condition (2.2). On the other hand, the set $\{\Phi_k\}_{k\ge1}$ is a orthonormal basis of $\mathbb{X}$. Thus, System (6.1) can be written as (2.3) where

$$\mathcal{A} := -A_0 = -\sum_{k\ge1} \mu_k (\cdot,\Phi_k)_{L^2(0,\pi)} \Phi_k\,,$$

and $\mathcal{B} \in \mathcal{L}(\mathbb{R}, H^{-1}(0,\pi))$ is given by

$$\mathcal{B} : v \in \mathbb{R} \longmapsto \mathcal{B}v = f(\cdot)v \in H^{-1}(0,\pi).$$

Evidently, $\mathcal{B}^* \in \mathcal{L}(H_0^1(0,\pi), \mathbb{R}) = H^{-1}(0,\pi)$ is given by

$$\langle \mathcal{B}^*, \phi \rangle_{H^{-1},H_0^1} = \langle f, \phi \rangle_{H^{-1},H_0^1}, \quad \forall \phi \in H_0^1(0,\pi).$$

With the previous notation and taking into account Proposition 6.1, it is easy to check that the operator $\mathcal{A}$ is the generator of a C^0–semigroup on $\mathbb{X}$ and $\mathcal{B} \in \mathcal{L}(\mathbb{R}, H^{-1}(0,\pi))$ is an admissible control operator for this semigroup. As a consequence, Corollary 2.4 can be applied obtaining the following approximate controllability result of System (6.1) in $L^2(0,\pi)$ at time T:

Proposition 6.2. *Under the previous assumptions, System* (6.1) *is approximately controllable in* $L^2(0,\pi)$ *at time* $T > 0$ *if and only if*

$$b_k = \langle f, \Phi_k \rangle_{H^{-1},H_0^1} \neq 0, \quad \forall k \geq 1. \tag{6.2}$$

Proof. The proof is a direct consequence of the first point of Corollary 2.4. The details are left to the reader. □

Let us now analyze the null controllability property of System (6.1) at time $T > 0$. The objective will be to apply Theorem 2.6 to this problem and, in particular, to determine the optimal time T_0 (see (2.13)) associated with the sequence Λ and the coefficients $\{b_k\}_{k\geq 1}$ (assuming that these coefficients satisfy (6.2)).

Firstly, the sequence $\Lambda = \{\mu_k\}_{k\geq 1} = \{k^2\}_{k\geq 1}$ satisfies the condition (2.2). In fact, this sequence also fulfills the property

$$|k^2 - l^2| \geq 3|k - l|, \quad \forall k, l \geq 1.$$

So that, Proposition 3.11 can be applied, getting $c(\Lambda) = 0$.

On the other hand, the direct application of Theorem 4.8 provides us the formula

$$T_0 = T_1 := \limsup \frac{\log \frac{1}{|b_k|}}{\mu_k} = \limsup \frac{-\log |b_k|}{k^2}, \tag{6.3}$$

where the coefficients $\{b_k\}_{k\geq 1}$ satisfy (6.2).

Summarizing, we can apply Theorem 2.6 and infer the following

Theorem 6.3. *Let us assume that* $f \in H^{-1}(0,\pi)$ *satisfies condition* (6.2) *and let us consider* $T_1 \in [0,\infty]$ *given by* (6.3)*. Then,*

1. *System* (6.1) *is null controllable in* $\mathbb{X} = L^2(0,\pi)$ *at any time* $T > T_1$.
2. *System* (6.1) *is not null controllable in* $\mathbb{X}$ *for* $T < T_1$. □

As a consequence of Theorems 6.2 and 6.3, let us study the case $f \equiv \delta_{x_0} \in H^{-1}(0,\pi)$ with $x_0 \in (0,\pi)$, i.e, the case in which we exert a pointwise control on the right hand side of the heat equation. In this case the coefficients b_k (see (6.2)) and the optimal time T_1 (see (6.3)) are given by

$$b_k = \Phi_k(x_0) = \sqrt{\frac{2}{\pi}} \sin kx_0, \; k \geq 1, \quad \text{and} \quad T_1 = \limsup \frac{-\log |\sin kx_0|}{k^2}.$$

Thus:

Corollary 6.4. *Assume that $f = \delta_{x_0} \in H^{-1}(0,\pi)$ with $x_0 \in (0,\pi)$. Under the previous notations, one has:*

1. *System (6.1) is approximately controllable in $L^2(0,\pi)$ at time $T > 0$ if and only if $x_0 \neq q\pi$ with $q \in \mathbb{Q} \cap (0,1)$.*

2. *Assume that $x_0 = \vartheta\pi$, with $\vartheta \in (0,1)$ an irrational number, and consider*

$$T_\vartheta = \limsup \frac{-\log|\sin(k\vartheta\pi)|}{k^2}.$$

   Then:

   (a) System (6.1) is null controllable in $L^2(0,\pi)$ at any time $T > T_\vartheta$.

   (b) System (6.1) is not null controllable in $L^2(0,\pi)$ for $T < T_\vartheta$. $\square$

Corollary 6.4 was proved by S. Dolecki in [9]. In fact, in that paper, the author proves some interesting result on the dependence of the optimal time T_ϑ with respect to ϑ:

Theorem 6.5 ([9])**.** *Under assumptions of Corollary 6.4, one has:*

1. *$T_\vartheta = 0$ for almost all $\vartheta \in [0,1]$.*

2. *Given $\tau \in [0,\infty]$, the set $\{\vartheta \in [0,1] : T_\vartheta = \tau\}$ is dense in $[0,1]$.* $\square$

Remark 6.6. The previous result shows the unstable dependence of T_ϑ with respect to $x_0 = \vartheta\pi$. Some similar results will be obtained in Subsection 6.2 for a non-scalar parabolic problem. $\square$

6.2 A boundary controllability problem for a non-scalar system

Let us now consider the one-dimensional controlled parabolic system

$$\begin{cases} \partial_t y - (D\partial_{xx} + A)\, y = 0 & \text{in } Q = (0,\pi)\times(0,T), \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases} \tag{6.4}$$

where $T > 0$ is a given time,

$$D = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \text{ (with } d > 0), \quad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \tag{6.5}$$

are given real constant matrices and $y_0 \in H^{-1}(0,\pi;\mathbb{R}^2)$ is the initial datum. Observe that $v \in L^2(0,T)$ is a scalar boundary control which acts on the Dirichlet boundary condition of the state at point $x = 0$ by means of the vector B. The objective is to control the whole system (two states) with one control force v.

Firstly, the previous problem is well-posed and a solution of (6.4) can be defined using for example the transposition method. Thus, one has:

Proposition 6.7. *Let us consider D given in (6.5) and $A \in \mathcal{L}(\mathbb{R}^2)$ and $B \in \mathbb{R}^2$. Then, for any $y_0 \in H^{-1}(0,\pi;\mathbb{R}^2)$ and $v \in L^2(0,T)$, System (6.4) admits a unique weak solution y satisfying $y \in L^2(Q;\mathbb{R}^2) \cap C^0([0,T];H^{-1}(0,\pi;\mathbb{R}^2))$ and*

$$\|y\|_{L^2(Q;\mathbb{R}^2)} + \|y\|_{C^0([0,T];H^{-1}(0,\pi;\mathbb{R}^2))} \le C\left(\|y_0\|_{H^{-1}(0,\pi;\mathbb{R}^2)} + \|v\|_{L^2(0,T)}\right),$$

where C is a positive constant only depending on D, A and B. $\square$

For a proof of the previous result see for instance [12] or [29].

The null and approximate controllability problems for System (6.4) has been studied mainly in the case $d = 1$ (see for instance [12] and [3]). In this case, the null and approximate controllability properties for System (6.4) are well-known and equivalent to an appropiate Kalman condition associated with (6.4). The general case $d > 0$ and $d \neq 1$ is more intricate and only a few results are known.

One has the following result:

Theorem 6.8 ([12], [3], [25]). *Let us consider System (6.4) with D given in (6.5). Then,*

1. *When $d = 1$, $A \in \mathcal{L}(\mathbb{R}^2)$ and $B \in \mathbb{R}^2$, System (6.4) is approximately controllable in $H^{-1}(0,\pi;\mathbb{R}^2)$ at time T if and only if it is exactly controllable to trajectories in $H^{-1}(0,\pi;\mathbb{R}^2)$ at time T if and only if*

$$\operatorname{rank}[B \,|\, AB] = 2 \quad \textit{and} \quad \Lambda_1 - \Lambda_2 \neq j^2 - k^2 \quad \forall k, j \in \mathbb{N} \textit{ with } k \neq j,$$

where Λ_1 and Λ_2 are the eigenvalues of A.

2. *When $d \neq 1$ and A and B are given in (6.5) with $b_1 = 0$ and $b_2 = 1$, System (6.4) is approximately controllable in $H^{-1}(0,\pi;\mathbb{R}^2)$ at time T if and only if $\sqrt{d} \notin \mathbb{Q}$.*

3. *There exists $d \in (0,\infty)$ with $\sqrt{d} \notin \mathbb{Q}$ such that System (6.4) is not null controllable at any time $T > 0$.* □

To our knowledge and apart from the previous result, the controllability properties of System (6.4) are completely open in the case $d \neq 1$. Let us then study the controllability of (6.4) in this case, i.e., when

$$d \neq 1.$$

Remark 6.9. It is interesting to point out that the second and third points of Theorem 6.8 shows that the controllability properties of System (6.4) seem to be very different from the corresponding controllability properties for scalar parabolic problems: In the non-scalar parabolic case (6.4) there are values $d > 0$ for which System (6.4) is approximate controllable at all positive time T and not null controllable at any time $T > 0$. □

Our objective is to apply Theorem 2.6 to System (6.4) and, to this end, let us first write this system under the abstract form (2.3).

Another way to define the solution to System (6.4), that will be adopted here, is making use of the notion of boundary control system as it is developed in [29, Chap. 10]. The self-adjoint operator

$$A_0 = -\frac{d^2}{dx^2} : L^2(0,\pi) \longrightarrow L^2(0,\pi)$$

with domain $D(A_0) = H^2(0,\pi) \cap H_0^1(0,\pi)$, admits various extensions. It is also a self-adjoint operator on $H^{-1}(0,\pi)$ with domain $H_0^1(0,\pi)$ and also on $\left(H^2(0,\pi) \cap H_0^1(0,\pi)\right)'$ with domain $L^2(0,\pi)$. Let us denote by A_0 all these extensions and let us work in

$$Z = H_0^1(0,\pi) + \mathcal{D}\mathbb{R} \subset H^1(0,\pi),$$

where $\mathcal{D}$ is the Dirichlet map: for each $v \in \mathbb{R}$, $z = \mathcal{D}v$ is the solution to the problem

$$\begin{cases} -z'' = 0 \quad \text{on } (0,\pi), \\ z(0) = v, \quad z(\pi) = 0, \end{cases}$$

i.e., $z(x) = \theta(x)v$ with $\theta(x) = (\pi - x)/\pi$. Let us also consider the differential operator $L \in \mathcal{L}(Z, H^{-1}(0,\pi))$ given by

$$Lz = -\frac{d^2 z}{dx^2}, \quad \forall z \in Z,$$

and the trace operator $\gamma_0 \in \mathcal{L}(H^1(0,\pi), \mathbb{R})$ defined by $\gamma_0 : z \in H^1(0,\pi) \mapsto \gamma_0 z = z(0) \in \mathbb{R}$. Observe that $L|_{H_0^1(0,\pi)} = A_0$. Thus, since $z - \mathcal{D}\gamma_0 z = z - \theta\gamma_0 z \in H_0^1(0,\pi)$ for any $z \in Z$, one has

$$A_0 \left(z - \mathcal{D}\gamma_0 z\right) = L\left(z - \mathcal{D}\gamma_0 z\right) = Lz,$$

i.e.,

$$L = A_0 - A_0\mathcal{D}\gamma_0.$$

With the previous decomposition of the operator L in the space Z, System (6.4) can be written under the form

$$\begin{cases} y' = \mathcal{A}y + \mathcal{B}v \quad \text{on } (0,T), \\ y(0) = y_0 \in \mathbb{X}, \end{cases} \tag{6.6}$$

where $\mathbb{X} = H^{-1}(0,\pi;\mathbb{R}^2)$,

$$\mathcal{A} := -DA_0 + A : \mathbb{X} \longrightarrow \mathbb{X}, \tag{6.7}$$

with $D(\mathcal{A}) = D(A_0) \times D(A_0) = H_0^1(0,\pi;\mathbb{R}^2)$, and $\mathcal{B} \in \mathcal{L}\Big(\mathbb{R}, \left(H^2(0,\pi;\mathbb{R}^2) \cap H_0^1(0,\pi;\mathbb{R}^2)\right)'\Big)$ is given by

$$\mathcal{B}v := DA_0\mathcal{D}Bv = (A_0\theta)DBv. \tag{6.8}$$

Let us observe that $\mathbb{X}$ is a Hilbert space for the scalar product

$$(y_1, y_2)_{\mathbb{X}} = \sum_{k\geq 1} \frac{1}{\mu_k}(y_{1,k}, y_{2,k})_{\mathbb{R}^2}, \quad \forall y_1, y_2 \in \mathbb{X},$$

where, as before, μ_k and Φ_k, $k \geq 1$, are, respectively, the eigenvalues and the eigenvectors of the Dirichlet laplacian in $(0,\pi)$ (i.e., of A_0, see (1.3)) and where $y_{i,k} \in \mathbb{R}^2$ is given by

$$y_{i,k} = \langle y_i, \Phi_k \rangle_{H^{-1}, H_0^1}, \quad \forall k \geq 1, \quad i = 1, 2.$$

Remark 6.10. We have reformulated System (6.4) under the abstract form (6.6). But in fact, the solution $y \in L^2(Q)$ to System (6.4) (and then, the solution to (6.6)) can be explicitly computed using the basis $\{\Phi_k\}_{k\geq 1}$ of $H^{-1}(0,\pi)$. Indeed, if $y_{0,k} = \langle y_0, \Phi_k \rangle_{H^{-1}, H_0^1} \in \mathbb{R}^2$, then,

$$y(t) = \sum_{k\geq 1} y_k(t)\Phi_k, \quad \text{a.e. } t \in (0,T),$$

where y_k is the solution to the ordinary differential system

$$\begin{cases} y_k' = (-k^2 D + A)y_k + k\sqrt{\dfrac{2}{\pi}}DBv \quad \text{on } (0,T), \\ y_k(0) = y_{0,k} \in \mathbb{R}^2. \end{cases} \tag{6.9}$$

One has:

Proposition 6.11. *Assume that $d \neq 1$. Let us consider the operators $\mathcal{A}$ and $\mathcal{B}$ given by (6.7) and (6.8) with D, A and B given by (6.5). Then, $\mathcal{A}$ is the generator of a C^0–semigroup on $\mathbb{X}$ and $\mathcal{B} \in \mathcal{L}\left(\mathbb{R}, \left(H^2(0,\pi;\mathbb{R}^2) \cap H_0^1(0,\pi;\mathbb{R}^2)\right)'\right)$ is an admissible control operator for this semigroup. Moreover,*

$$\sigma(\mathcal{A}) = \left\{-k^2, -dk^2\right\}_{k\geq 1} := \{-\lambda_{1,k}, -\lambda_{2,k}\}_{k\geq 1} \tag{6.10}$$

and the corresponding family of eigenfunctions is given by

$$\{\phi_{1,k}, \phi_{2,k}\}_{k\geq 1} := \{kV_{1,k}\Phi_k, kV_{2,k}\Phi_k\}_{k\geq 1}$$

where $V_{i,k} \in \mathbb{R}^2$ are such that $(-k^2 D + A)V_{i,k} = -\lambda_{i,k}V_{i,k}$, for any $k \geq 1$ and $i = 1, 2$, i.e.,

$$V_{1,k} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad V_{2,k} = \begin{pmatrix} -\frac{1}{(d-1)k^2} \\ 1 \end{pmatrix}, \quad \forall k \geq 1.$$

Finally, the eigenvalues of $\mathcal{A}$ are simple, i.e., $-\lambda_{1,k} \neq -\lambda_{2,j}$ for any $k, j \geq 1$, if and only if $\sqrt{d} \notin \mathbb{Q}$.

Proof. That $\mathcal{A}$ is the generator of a C^0–semigroup on $\mathbb{X}$ can be checked by showing that $\mathcal{A} + \alpha I_d$ is a maximal monotone operator on $\mathbb{X}$ for $\alpha > 0$ large enough. On the other hand, applying Proposition 6.7 we deduce that System (6.6) possesses a unique solution $y \in C^0([0,T];\mathbb{X})$ for $y_0 = 0$ and for any $v \in L^2(0,T)$. In particular, $y(T) \in \mathbb{X}$ and this proves that $R(L_T) \subset \mathbb{X}$ (L_T is given by (2.6)). Then, we have that $\mathcal{B} \in \mathcal{L}\left(\mathbb{R}, \left(H^2(0,\pi;\mathbb{R}^2) \cap H_0^1(0,\pi;\mathbb{R}^2)\right)'\right)$ is an admissible control operator for the semigroup associated with $\mathcal{A}$.

Clearly, $\mathcal{A}$ has compact resolvent and it is easy to check that its (point) spectrum is given by (6.10). The remainder of the statement can be also easily checked and the details will be omitted. □

In the sequel, A^* (resp., B^*) will denote the transpose of the matrix A (resp., of the vector B).

We continue checking the assumptions of Theorem 2.6 applied to System (6.4). One has:

Proposition 6.12. *Under the hypotheses of Proposition 6.11, the following properties hold:*

1. *The family $\{\phi_{1,k}, \phi_{2,k}\}_{k\geq 1}$ is a Riesz basis of eigenfunctions of $\mathcal{A}$ in $\mathbb{X}$. Its biorthogonal basis is given by*

$$\{\psi_{1,k}, \psi_{2,k}\}_{k\geq 1} := \{kW_{1,k}\Phi_k, kW_{2,k}\Phi_k\}_{k\geq 1}$$

where $W_{i,k} \in \mathbb{R}^2$ are such that $(-k^2 D + A^*)W_{i,k} = -\lambda_{i,k}W_{i,k}$, for any $k \geq 1$ and $i = 1, 2$, i.e.,

$$W_{1,k} = \begin{pmatrix} 1 \\ \frac{1}{(d-1)k^2} \end{pmatrix}, \quad W_{2,k} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \forall k \geq 1.$$

2. *The operator $\mathcal{A}$ can be written as*

$$\mathcal{A} = -\sum_{k\geq 1} \left[\lambda_{1,k}(\cdot, \psi_{1,k})_{\mathbb{X}}\phi_{1,k} + \lambda_{2,k}(\cdot, \psi_{2,k})_{\mathbb{X}}\phi_{2,k}\right].$$

3. *The operator $\mathcal{B}^* \in \mathcal{L}\left(H^2(0,\pi;\mathbb{R}^2) \cap H_0^1(0,\pi;\mathbb{R}^2), \mathbb{R}\right) \equiv \left(H^2(0,\pi;\mathbb{R}^2) \cap H_0^1(0,\pi;\mathbb{R}^2)\right)'$ satisfies*

$$\langle \mathcal{B}^*, \psi_{i,k}\rangle = \sqrt{\frac{2}{\pi}} B^* D W_{i,k}, \quad \forall k \geq 1, \quad i = 1, 2, \tag{6.11}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between the spaces $\left(H^2(0,\pi;\mathbb{R}^2) \cap H_0^1(0,\pi;\mathbb{R}^2)\right)'$ and $H^2(0,\pi;\mathbb{R}^2) \cap H_0^1(0,\pi;\mathbb{R}^2)$.

Proof. It is straightforward to prove that the family $\{\phi_{1,k}, \phi_{2,k}\}_{k\geq 1}$ is complete in $\mathbb{X}$. On the other hand, it is also easy to see that $\{\psi_{1,k}, \psi_{2,k}\}_{k\geq 1}$ is its biorthogonal family, i.e.,

$$(\phi_{i,k}, \psi_{j,\ell})_{\mathbb{X}} = \delta_{ij}\delta_{k\ell}, \quad \forall k, \ell \geq 1, \quad i, j \in \{1, 2\}.$$

This last property in particular implies that both families form strongly independent sets, i.e.,

$$\phi_{j,\ell} \notin \overline{\text{span}}\,\{\phi_{i,k} : (i,k) \neq (j,\ell)\} \text{ and } \psi_{j,\ell} \notin \overline{\text{span}}\,\{\psi_{i,k} : (i,k) \neq (j,\ell)\}, \quad \forall \ell \geq 1,\ j = 1, 2.$$

In order to see that $\{\phi_{1,k}, \phi_{2,k}\}_{k\geq 1}$ is a Riesz basis in $\mathbb{X}$, we will use the following result (see for instance [14, p. 320]):

Lemma 6.13. *Let $\{x_k\}_{k\geq 1}$ be a sequence in a Hilbert space X. Then the following statements are equivalent.*

(a) $\{x_k\}_{k\geq 1}$ *is a Riesz basis in X.*

(b) $\{x_k\}_{k\geq 1}$ *is a complete Bessel sequence in X and possesses a biorthogonal system $\{y_k\}_{k\geq 1}$ that is also a complete Bessel sequence in X.* □

We recall that the sequence $\{x_k\}_{k\geq 1}$ in the Hilbert space X is a Bessel sequence if it satisfies

$$\sum_{k\geq 1} |(x, x_k)_X|^2 < \infty, \quad \forall x \in X.$$

Using the previous result, we only have to prove that $\{\phi_{1,k}, \phi_{2,k}\}_{k\geq 1}$ and $\{\psi_{1,k}, \psi_{2,k}\}_{k\geq 1}$ are Bessel sequences in $\mathbb{X}$.

Let us fix $f \in \mathbb{X}$, i.e., $f = (f_1, f_2)$ with $f_1, f_2 \in H^{-1}(0, \pi)$. Thus, for any $k \geq 1$,

$$\begin{cases} |(f, \phi_{1,k})_{\mathbb{X}}| = \dfrac{1}{k} \left| \langle f_1, \Phi_k \rangle_{H^{-1}, H_0^1} \right| = \dfrac{1}{k} |f_{1,k}|, \\ |(f, \phi_{2,k})_{\mathbb{X}}| = \dfrac{1}{k} \left| \langle f, V_{2,k}\Phi_k \rangle_{H^{-1}, H_0^1} \right| = \dfrac{1}{k} \left| \dfrac{-f_{1,k}}{(d-1)k^2} + f_{2,k} \right|, \end{cases}$$

and therefore,

$$\sum_{k\geq 1} \left(|(f, \phi_{1,k})_{\mathbb{X}}|^2 + |(f, \phi_{2,k})_{\mathbb{X}}|^2 \right) \leq C \sum_{k\geq 1} \frac{1}{k^2} \left(|f_{1,k}|^2 + |f_{2,k}|^2 \right) < \infty,$$

with C a positive constant. This shows that $\{\phi_{1,k}, \phi_{2,k}\}_{k\geq 1}$ is a Bessel sequence in $\mathbb{X}$. A similar argument proves that $\{\psi_{1,k}, \psi_{2,k}\}_{k\geq 1}$ is also a Bessel sequence in $\mathbb{X}$. We have completed the first point of the result.

The second point is a consequence of the first point and can be easily checked.

For proving the third point, let us recall that $\theta(x) = \dfrac{\pi - x}{\pi}$ $(x \in (0, \pi))$ and note that

$$(\theta, \Phi_k)_{L^2(0,\pi)} = \sqrt{\frac{2}{\pi}} \int_0^\pi \frac{\pi - x}{\pi} \sin kx \, dx = \sqrt{\frac{2}{\pi}} \frac{1}{k}, \quad \forall k \geq 1.$$

Thus, using (6.8) and the previous equality, we get

$$\begin{aligned} \langle v\mathcal{B}^*, \psi_{i,k} \rangle = \langle \mathcal{B}v, \psi_{i,k} \rangle = \langle (A_0\theta) DBv, \psi_{i,k} \rangle &= \frac{1}{k} (DBv, W_{i,k})_{\mathbb{R}^2} \langle A_0\theta, \Phi_k \rangle \\ &= \sqrt{\frac{2}{\pi}} (B^* D W_{i,k})\, v, \quad \forall v \in \mathbb{R}, \quad \forall k \geq 1. \end{aligned}$$

This proves the third point and finalizes the proof. □

We are now ready to state the approximate controllability result for System (6.4). This result also provides the proof of the first point of Theorem 2.8. It reads as follows:

Theorem 6.14. *Let us suppose the assumptions of Proposition 6.11. Then, System (6.4) is approximately controllable in $\mathbb{X} = H^{-1}(0, \pi; \mathbb{R}^2)$ at time $T > 0$ if and only if*

$$\sqrt{d} \notin \mathbb{Q}, \tag{6.12}$$

$$b_2 \left[(d-1)\, k^2 b_1 + d b_2 \right] \neq 0, \quad \forall k \geq 1. \tag{6.13}$$

Proof. Let us first proof that (6.12) and (6.13) are necessary conditions for the approximate controllability of System (6.4) in $\mathbb{X}$ at time T. Indeed, if $\sqrt{d} \in \mathbb{Q}$ then, from Proposition 6.11, the operator $-\mathcal{A}$ has eigenvalues with geometric multiplicity equal to two: there exist $k, j \in \mathbb{N}$ such that $\lambda_{1,k} = \lambda_{2,j}$. In this case, it is easy to see that, taking

$$\varphi_0 = a_1 \psi_{1,k} + a_2 \psi_{2,j}, \quad \text{with} \quad a_1, a_2 \in \mathbb{R},$$

the solution φ to the corresponding adjoint problem (2.8) is given by (see (2.9)):

$$\varphi(t) = e^{-\widetilde{\lambda}(T-t)} \left(a_1 \psi_{1,k} + a_2 \psi_{2,j} \right), \quad \forall t \in (0, T),$$

with $\widetilde{\lambda} = \lambda_{1,k} = \lambda_{2,j}$. Now, using the identity (6.11) we get

$$\mathcal{B}^* \varphi(t) = \sqrt{\frac{2}{\pi}} e^{-\widetilde{\lambda}(T-t)} \left(a_1 B^* D W_{1,k} + a_2 B^* D W_{2,j} \right), \quad \forall t \in (0, T).$$

This last formula proves that there exist $a_1, a_2 \in \mathbb{R}$, with $a_1 a_2 \neq 0$, such that $\mathcal{B}^* \varphi(t) = 0$ for any $t \in (0, T)$. We can conclude that the adjoint problem to (6.4) does not satisfy the unique continuation property (see Theorem 2.3) and therefore, System (6.4) is not approximate controllable in $\mathbb{X}$ at time T.

Let us now assume that $d \neq 1$ fulfills condition (6.12). From Propositions 6.11 and 6.12 we deduce that the sequence $\Lambda = \{\lambda_{1,k}, \lambda_{2,k}\}_{k \geq 1}$ and the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy the assumptions of Corollary 2.4. Thus, we can directly apply the first point of this result and obtain that System (6.4) is approximately controllable in $\mathbb{X}$ at time T if and only if

$$b_{i,k} := \mathcal{B}^* \psi_{i,k} \neq 0, \quad \forall k \geq 1, \quad i = 1, 2.$$

From (6.11) we get (6.13). This ends the proof. □

Remark 6.15. Conditions (6.12) and (6.13) are independent of the final observation time $T > 0$ and can be seen as a Kalman condition for the approximate controllability of System (6.4) at time T. In fact, Condition (6.12) is equivalent to a condition on the simplicity of the eigenvalues of the operator $\mathcal{A}$ (given by (6.7)). On the other hand, condition (6.13) is equivalent to the controllability of the ordinary differential system (6.9) for any $k \geq 1$. Indeed, given $k \geq 1$, System (6.9) is controllable at time T if and only if (Kalman rank condition)

$$\text{rank} \left[DB \,|\, \left(-k^2 D + A \right) DB \right] = 2,$$

and this amounts to condition (6.13). □

Remark 6.16. Observe that when

$$B = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

condition (6.13) holds and the approximate controllability result for System (6.4) at time T is equivalent to condition (6.12). Thus, the previous result generalizes the approximate controllability result for System (6.4) given in [12], to the case in which B is a general vector in $\mathbb{R}^2$. □

We are now in position to set, as a consequence of Theorem 2.6, the null controllability result for System (6.4). This result in particular proves the second point of Theorem 2.8:

Theorem 6.17. *Let us consider the matrices D, A and B given by (6.5) with $d \neq 1$. Let $c(\Lambda_d)$ be the index of condensation of the sequence $\Lambda_d := \{\lambda_{1,k}, \lambda_{2,k}\}_{k\geq 1} = \left\{k^2, dk^2\right\}_{k\geq 1}$. Let us also assume that conditions (6.12) and (6.13) hold. Then,*

1. *System (6.4) is null controllable in $\mathbb{X} = H^{-1}(0,\pi;\mathbb{R}^2)$ at any time $T > c(\Lambda_d)$.*
2. *System (6.4) is not null controllable in $\mathbb{X}$ for $T < c(\Lambda_d)$.*

Proof. First, let us take the real sequence $\Lambda_d = \{\Lambda_k\}_{k\geq 1}$ normally ordered. The assumption (6.12) ensures that this sequence Λ_d of eigenvalues of the operator $-\mathcal{A}$ satisfies condition (2.2). On the other hand, from Propositions 6.11 and 6.12 we also deduce that the operators $\mathcal{A}$ and $\mathcal{B}$ fulfill the conditions of Theorem 2.6. Finally, condition (2.10) corresponds to (6.13). Indeed, using (6.11) we get

$$\begin{cases} b_{1,k} := \mathcal{B}^*\psi_{1,k} = B^*DW_{1,k} = b_1 + \frac{db_2}{(d-1)k^2}, \\ b_{2,k} := \mathcal{B}^*\psi_{2,k} = B^*DW_{2,k} = db_2, \end{cases}$$

for any $k \geq 1$. Thus we can apply Theorem 2.6 and obtain the result for T_0 given by (2.13). Let us now compute this optimal time T_0. We clearly have

$$\lim \frac{\log 1/|b_{i,k}|}{\lambda_{i,k}} = 0, \quad i = 1,2.$$

Thus, from Theorem 4.8, we infer $T_0 = c(\Lambda_d)$. This ends the proof. □

Remark 6.18. Let us observe that the assumption $\sqrt{d} \notin \mathbb{Q}$ guarantees that the eigenvalues of the operator $-\mathcal{A}$ (given by (6.7)) are simple and satisfies (2.2). In particular, we can compute the corresponding condensation index of the sequence Λ_d of eigenvalues of $-\mathcal{A}$ (see Remark 2.7) and this condensation index provides the optimal time T_0 for the null controllability of System (6.4). Evidently, this optimal time is independent of the control vector B and only depends on the diffusion coefficient d. □

Remark 6.19. In the following result we will see that for some diffusion coefficients d satisfying (6.12) one has $c(\Lambda_d) = 0$, where Λ_d is the sequence $\Lambda_d = \left\{k^2, dk^2\right\}_{k\geq 1}$. In that case, System (6.4) is approximate controllable in $H^{-1}(0,\pi;\mathbb{R}^2)$ at time $T > 0$ if and only if the system is null controllable in this space at time T and these two properties are equivalent to condition (6.13).

We will also see that there exists $d > 0$ satisfying (6.12) for which $c(\Lambda_d) > 0$ or $c(\Lambda_d) = \infty$. In these cases the approximate controllability property at time T for System (6.4) is not equivalent to the null controllability one at time T. The extreme situation is reached in then case $c(\Lambda_d) = \infty$ and $B \in \mathbb{R}^2$ satisfying (6.13): The system is approximate controllable at every time $T > 0$ and never null controllable at any positive time. □

From Theorem 6.17 we deduce that System (6.4) has a null controllability optimal time $T_0 = c(\Lambda_d)$ which strongly depends on the diffusion matrix D. At this level, a natural question arises: given $\tau_0 \in [0,\infty]$, does there exist $d > 0$ satisfying (6.12) and such that $c(\Lambda_d) = \tau_0$? The answer is positive and is given by the following proposition (which is, in fact, the first point of Theorem 2.9):

Proposition 6.20. *For any $\tau_0 \in [0,\infty]$, there exists $d \in (0,\infty)$ satisfying (6.12) such that the condensation index of the sequence $\Lambda_d = \{k^2, dk^2\}_{k\geq 1}$ is given by $c(\Lambda_d) = \tau_0$.*

Proof. Let us fix $\tau_0 \in [0,\infty]$. The objective is to determine $d > 0$ such that the condensation index associated with Λ_d (see (3.2)) is equal to τ_0. To this end, let us consider the infinite product associated with the sequence Λ_d:

$$E\left(z\right) = \prod_{k\geq 1}\left(1-\frac{z^2}{k^4}\right)\left(1-\frac{z^2}{d^2k^4}\right),$$

and let us first calculate $|E'(k^2)|$ and $|E'(dk^2)|$ for any $k \geq 1$.

From Euler's formula

$$\sin(\pi z) = \pi z\prod_{k\geq 1}\left(1-\frac{z^2}{k^2}\right), \quad \forall z \in \mathbb{C},$$

we deduce the following expression:

$$E\left(\zeta^2\right) = d\frac{\sin\left(\pi\zeta\right)\sin\left(\frac{\pi\zeta}{\sqrt{d}}\right)\sinh\left(\pi\zeta\right)\sinh\left(\frac{\pi\zeta}{\sqrt{d}}\right)}{\pi^4\zeta^4}, \quad \forall \zeta \in \mathbb{C}.$$

Differentiating with respect to ζ, we can readily deduce:

$$\left|E'(k^2)\right| = \left|\sin\left(\frac{\pi k}{\sqrt{d}}\right)\right| A_k, \quad \left|E'(dk^2)\right| = \left|\sin\left(\pi k\sqrt{d}\right)\right| B_k, \quad \forall k \geq 1, \tag{6.14}$$

where

$$A_k := \frac{d}{2\pi^3k^5}\sinh\left(\pi k\right)\sinh\left(\frac{\pi k}{\sqrt{d}}\right), \quad B_k := \frac{1}{2\pi^3d^2k^5}\sinh\left(\pi k\right)\sinh\left(\pi k\sqrt{d}\right),$$

for any $k \geq 1$. Finally, it is easy to see the estimates

$$\begin{cases} \dfrac{d}{2\pi^3k^5}\sinh\left(\pi\right)\sinh\left(\dfrac{\pi}{\sqrt{d}}\right) \leq A_k \leq \dfrac{d}{8\pi^3k^5}e^{\pi k}e^{\pi k/\sqrt{d}}, \\ \dfrac{1}{2\pi^3d^2k^5}\sinh\left(\pi\right)\sinh\left(\pi\sqrt{d}\right) \leq B_k \leq \dfrac{1}{8\pi^3d^2k^5}e^{\pi k}e^{\pi k\sqrt{d}}, \quad \forall k \geq 1. \end{cases} \tag{6.15}$$

Let us observe that $c(\Lambda_d)$ is given by (3.2) with $\Lambda_d = \{\lambda_k\}_{k\geq 1} := \{k^2, dk^2\}_{k\geq 1}$. Therefore, $c(\Lambda_d) = \max\{l_1, l_2\}$ with

$$l_1 = \limsup \frac{\log\frac{1}{|E'(k^2)|}}{k^2} \quad \text{and} \quad l_2 = \limsup \frac{\log\frac{1}{|E'(dk^2)|}}{dk^2}.$$

Now, using the expression (6.14) and the estimates (6.15), we also have $c(\Lambda_d) = \max\{l_1, l_2\}$ with

$$l_1 = \limsup \frac{-\log\left|\sin\left(\frac{\pi k}{\sqrt{d}}\right)\right|}{k^2} \quad \text{and} \quad l_2 = \limsup \frac{-\log\left|\sin\left(\pi k\sqrt{d}\right)\right|}{dk^2}. \tag{6.16}$$

Case $\tau_0 = 0$. In order to prove the result in the case $\tau_0 = 0$, let us recall a well-known lemma about approximation of algebraic numbers:

Lemma 6.21. *Let ν be an irrational algebraic number of degree $n \geq 2$, i.e., ν is an irrational number which is the root of a polynomial of degree n with integer coefficients. Then, there exists a positive number C, depending on ν, such that*

$$\left| \nu - \frac{p}{q} \right| > \frac{C}{q^n}, \quad \forall p, q \in \mathbb{N}, \quad q > 0. \qquad \square$$

The previous result is known as Liouville's theorem on diophantine approximation. For a proof, see for instance [22].

Let us take $d > 0$ such that $\sqrt{d}$ is an irrational algebraic number of degree $n \geq 2$ and let us see that $c(\Lambda_d) = 0$.

We reasoning as follows. For any $k \geq 1$ there exists $h_k \in \mathbb{N}$ such that

$$\left| k\sqrt{d} - h_k \right| \leq \frac{1}{2}. \tag{6.17}$$

Indeed, we can take $h_k = \left\lfloor k\sqrt{d} \right\rfloor$ if $k\sqrt{d} - \left\lfloor k\sqrt{d} \right\rfloor \leq 1/2$ or $h_k = \left\lfloor k\sqrt{d} \right\rfloor + 1$ otherwise ($\lfloor \cdot \rfloor$ is the floor function, i.e., for $x \in \mathbb{R}$, $\lfloor x \rfloor$ gives the largest integer less than or equal to x).

If we now apply Lemma 6.21 with $q = k$ and $p = h_k$ we get

$$\frac{C}{k^{n-1}} \leq \left| k\sqrt{d} - h_k \right| \leq \frac{1}{2}, \quad \forall k \geq 1,$$

and

$$\left| \sin \left[\pi \left(k\sqrt{d} - h_k \right) \right] \right| = \sin \left| \pi \left(k\sqrt{d} - h_k \right) \right| \geq \sin \left| C\pi k^{1-n} \right| = \left| \sin \left(C\pi k^{1-n} \right) \right|, \quad \forall k \geq 1.$$

Recall that $c(\Lambda_d) = \max\{l_1, l_2\}$ with l_1 and l_2 given in (6.16). So,

$$\begin{aligned} l_2 &= \limsup \frac{-\log \left| \sin \left(\pi k\sqrt{d} \right) \right|}{dk^2} = \limsup \frac{-\log \left| \sin \left(\pi k\sqrt{d} - \pi h_k \right) \right|}{dk^2} \\ &\leq \limsup \frac{-\log \left| \sin \left(C\pi k^{1-n} \right) \right|}{dk^2} = 0. \end{aligned}$$

The same argument applied to $1/\sqrt{d}$ permits to prove that $l_1 \leq 0$. Taking into account Remark 3.10, we deduce that $c(\Lambda_d) = 0$.

Case $\tau_0 \in (0, \infty)$. Let us now show the result when $\tau_0 \in (0, \infty)$. We will use the following

Lemma 6.22. *1. For any $\tau_0 \in (0, \infty)$, there exist an irrational number $d > 0$ and a sequence of rational numbers $\{p_k/q_k\}_{k \geq 0}$ such that p_k and q_k are co-prime positive integers, the sequences $\{p_k\}_{k \geq 0}$ and $\{q_k\}_{k \geq 0}$ are strictly increasing and*

$$\lim e^{\tau_0 p_k^2} \left| \sqrt{d} - \frac{p_k}{q_k} \right| = 1. \tag{6.18}$$

Moreover, for any $k \geq 0$ one has

$$\left| q_k\sqrt{d} - p_k \right| \leq \left| q\sqrt{d} - p \right|, \quad \forall p, q \in \mathbb{N}, \quad \textit{with } q \leq q_k. \tag{6.19}$$

2. *For any $\sigma \in (0,\infty)$, there exists an irrational number $d > 0$ and a sequence of rational numbers $\{p_k/q_k\}_{k\geq 0}$ such that p_k and q_k are co-prime positive integers, the sequences $\{p_k\}_{k\geq 0}$ and $\{q_k\}_{k\geq 0}$ are strictly increasing and*

$$\lim e^{p_k^{2+\sigma}} \left|\sqrt{d} - \frac{p_k}{q_k}\right| = 0. \tag{6.20}$$

The proof of this lemma is based on some properties of continued fractions. The second point follows some ideas from [25]. We will give its proof in Appendix A.

Let us fix $\tau_0 \in (0,\infty)$ and take $d > 0$ provided by the first point of Lemma 6.22. Again, recall that $c(\Lambda_d) = \max\{l_1, l_2\}$ with l_1 and l_2 given by (6.16). The aim is now to prove that $c(\Lambda_d) = \tau_0$ and, to this end, we will show that $l_1 = l_2 = \tau_0$. We will divide the proof into two steps.

A. Observe that

$$\frac{-\log\left|\sin\left(\frac{\pi k}{\sqrt{d}}\right)\right|}{k^2} = \frac{-\log\left|\sin\left(\frac{\pi}{\sqrt{d}}\left(k - h\sqrt{d}\right)\right)\right|}{k^2}, \quad \forall h \in \mathbb{N}.$$

Let us now take the subsequences of positive integers $\{p_k\}_{k\geq 1}$ and $\{q_k\}_{k\geq 1}$ provided by Lemma 6.22. From (6.18) we deduce $\lim\left(p_k - q_k\sqrt{d}\right) = 0$, $\lim p_k/q_k = \sqrt{d}$ and

$$\begin{aligned} l_1 &\geq \limsup \frac{-\log\left|\sin\left(\frac{\pi}{\sqrt{d}}\left(p_k - q_k\sqrt{d}\right)\right)\right|}{p_k^2} = \limsup \frac{-\log\left|\frac{\pi}{\sqrt{d}}\left(p_k - q_k\sqrt{d}\right)\right|}{p_k^2} \\ &= \lim \frac{-\log\left|\frac{\pi}{\sqrt{d}} q_k e^{-\tau_0 p_k^2}\right|}{p_k^2} = \tau_0. \end{aligned}$$

Let us now reason with l_2. As before, from (6.18) we get

$$\begin{aligned} l_2 &\geq \limsup \frac{-\log\left|\sin\left(\pi q_k\sqrt{d}\right)\right|}{d q_k^2} = \limsup \frac{-\log\left|\sin\left(\pi q_k\sqrt{d} - \pi p_k\right)\right|}{d q_k^2} \\ &= \limsup \frac{-\log\left|\pi q_k\sqrt{d} - \pi p_k\right|}{d q_k^2} = \limsup \frac{-\log\left|\pi q_k e^{-\tau_0 p_k^2}\right|}{d q_k^2} = \lim \frac{\tau_0 p_k^2}{d q_k^2} = \tau_0. \end{aligned}$$

In conclusion, we have obtained that $l_1 \geq \tau_0$ and $l_2 \geq \tau_0$. Using once more that $c(\Lambda_d) = \max\{l_1, l_2\}$, we can also conclude $c(\Lambda_d) \geq \tau_0$.

B. Let us now see the inequalities $l_1 \leq \tau_0$ and $l_2 \leq \tau_0$. As before, for each $k \geq 1$ there exists $h_k \in \mathbb{N}$ such that (6.17) holds. On the other hand, there exists $n_k \in \mathbb{N}$ such that $k \leq q_{n_k}$. Since every convergent p_k/q_k satisfies (6.19), it follows that:

$$\left|k\sqrt{d} - h_k\right| \geq \left|q_{n_k}\sqrt{d} - p_{n_k}\right|.$$

From this last inequality and (6.17), we deduce:

$$\begin{cases} \left|\sin\left(\pi k\sqrt{d}\right)\right| = \left|\sin\left[\pi\left(k\sqrt{d} - h_k\right)\right]\right| = \sin\left|\pi\left(k\sqrt{d} - h_k\right)\right| \geq \sin\left|\pi\left(q_{n_k}\sqrt{d} - p_{n_k}\right)\right| \\ \qquad = \left|\sin\left[\pi\left(q_{n_k}\sqrt{d} - p_{n_k}\right)\right]\right|, \end{cases}$$

for any $k \geq 1$. Thus, since $k \leq q_{n_k}$,

$$\frac{\log\left|\sin\left(\pi k\sqrt{d}\right)\right|}{dk^2} \geq \frac{\log\left|\sin\left[\pi\left(q_{n_k}\sqrt{d}-p_{n_k}\right)\right]\right|}{dq_{n_k}^2}.$$

Coming back to the expression of l_2 (see (6.16)), we obtain

$$l_2 = \limsup \frac{-\log\left|\sin\left(\pi k\sqrt{d}\right)\right|}{dk^2} \leq \limsup \frac{-\log\left|\sin\left[\pi\left(q_{n_k}\sqrt{d}-p_{n_k}\right)\right]\right|}{dq_{n_k}^2} = \tau_0.$$

In the previous inequality we have used (6.18), $\lim\left(p_{n_k} - q_{n_k}\sqrt{d}\right) = 0$ and $\lim p_{n_k}/q_{n_k} = \sqrt{d}$.

A similar argument permits to prove the inequality $l_1 \leq \tau_0$. In conclusion, we have proved that $l_1 = l_2 = \tau_0$ and, therefore, the existence of an irrational number $d > 0$ such that $c(\Lambda_d) = \tau_0 \in (0, \infty)$.

Case $\tau_0 = \infty$. In order to get the result in this case, let us fix $\sigma > 0$ and apply the second point of Lemma 6.22. We deduce the existence of an irrational number $d > 0$ and a sequence of rational numbers $\{p_k/q_k)_{k\geq 0}$ satisfying (6.20). In particular, we deduce the existence of a positive constant C such that

$$|q_k\sqrt{d} - p_k| \leq Cq_k e^{-p_k^{2+\sigma}}, \quad \forall k \geq 1.$$

Following previous arguments, it is not difficult to see

$$\begin{aligned} l_1 &\geq \limsup \frac{-\log\left|\sin\left(\frac{\pi}{\sqrt{d}}\left(p_k - q_k\sqrt{d}\right)\right)\right|}{p_k^2} = \limsup \frac{-\log\left|\frac{\pi}{\sqrt{d}}\left(p_k - q_k\sqrt{d}\right)\right|}{p_k^2} \\ &\geq \lim \frac{-\log\left|\frac{C\pi}{\sqrt{d}} q_k e^{-p_k^{2+\sigma}}\right|}{p_k^2} = \infty. \end{aligned}$$

This proves the third case and finalizes the proof of the result. □

Remark 6.23. As a consequence of Proposition 6.20 we deduce that if $\sqrt{d}$ is an irrational algebraic number then $c(\Lambda_d) = 0$, where Λ_d is the sequence $\Lambda_d = \{k^2, dk^2\}_{k\geq 1}$. This condition on d can be easily generalized to the case in which d satisfies the property

$$\left|\sqrt{d} - \frac{p}{q}\right| \geq \frac{\widetilde{\Phi}_1(q)}{q}, \quad \forall p, q \in \mathbb{N}, \quad p, q \geq k_0,$$

where $k_0 \in \mathbb{N}$ and $\widetilde{\Phi}_1$ is a positive function that fulfills the conditions

$$\lim \widetilde{\Phi}_1(k) = 0 \quad \text{and} \quad \limsup \frac{\log\left(\widetilde{\Phi}_1(k)\right)}{k^2} = 0.$$

This condition can be equivalently written as follows: *"Let $\widetilde{\Phi}_1$ be a positive function satisfying the previous conditions. If $d \in (0, \infty)$ is an irrational number such that the inequality*

$$\left|\sqrt{d} - \frac{p}{q}\right| < \frac{\widetilde{\Phi}_1(q)}{q}$$

has a finite number of integer solutions $p, q > 0$*, then* $c(\Lambda_d) = 0$*".*

On the other hand, it is possible to generalize condition (6.20) in this way: *"Let* $\widetilde{\Phi}_2$ *be a positive function satisfying*

$$\lim \frac{-\log\left(\widetilde{\Phi}_2(k)\right)}{k^2} = \infty.$$

If $d > 0$ *is an irrational number such that the inequality*

$$\left|\sqrt{d} - \frac{p}{q}\right| \leq \widetilde{\Phi}_2(p)$$

has an infinite number of integer solutions $p, q > 0$*, then* $c(\Lambda_d) = \infty$*".* □

We will finalize this subsection giving two results on the measure of irrationals numbers $\sqrt{d}$ for which the optimal time of null controllability of System (6.4), $T_0 = c(\Lambda_d)$, is zero or positive. In particular these results provide the proof of the two last points of Theorem 2.9.

Proposition 6.24. *Let* $c(\Lambda_d)$ *be the index of condensation of the sequence* $\Lambda_d := \left\{k^2, dk^2\right\}_{k\geq 1}$*, with* $\sqrt{d} \notin \mathbb{Q}$*. Then,* $c(\Lambda_d) = 0$ *for almost* $d \in (0, \infty)$*.*

Proof. The proof is a consequence of Remark 6.23 and Theorem 32 in [22] (p. 69). Indeed, let us consider the inequality

$$\left|\sqrt{d} - \frac{p}{q}\right| < \frac{\widetilde{\Phi}_1(q)}{q} \tag{6.21}$$

with $\widetilde{\Phi}_1(x) = 1/x^2$. Observe that

$$\int_1^\infty \widetilde{\Phi}_1(x)\, dx < \infty.$$

Thus, the inequality (6.21) has a finite number of integer solutions $p, q \in \mathbb{N}$ for any $d \in (0, \infty) \setminus \mathcal{M}$, with $|\mathcal{M}| = 0$. In particular, $c(\Lambda_d) = 0$ for any $d \in (0, \infty) \setminus \mathcal{M}$. This proves the result. □

Corollary 6.25. *Let* $c(\Lambda_d)$ *be the index of condensation of the sequence* $\Lambda_d := \left\{k^2, dk^2\right\}_{k\geq 1}$*, with* $\sqrt{d} \notin \mathbb{Q}$*. Then, given* $\tau_0 \in [0, \infty]$*, the set*

$$\{d \in (0, \infty) : c(\Lambda_d) = \tau_0\}$$

is dense in $(0, \infty)$*.* □

Let us observe that the previous result is clear for $\tau_0 = 0$. The case $\tau_0 \in (0, \infty]$ is a consequence of Lemma 6.22 and will be proved in Appendix A.

Remark 6.26. As said before, the optimal time for the null controllability of System (6.4) in the space $H^{-1}(0, \pi; \mathbb{R}^2)$ is given by $T_0 = c(\Lambda_d)$ with $\Lambda_d = \{k^2, dk^2\}_{k\geq 1}$. Evidently, this optimal time depends strongly on the diffusion coefficient d and depends on the diophantine approximation of the irrational number $\sqrt{d}$ by rational numbers. Observe also that for some $d \in (0, \infty)$ the optimal time is $T_0 = c(\Lambda_d) = \infty$. In this sense, Theorem 6.17 generalizes the results in [25]. □

Summarizing, we have proved (see Theorems 6.14 and 6.17):

1. If $\sqrt{d} \in \mathbb{Q}$, then System (6.4) is neither approximate nor null controllable in $H^{-1}(0, \pi; \mathbb{R}^2)$ at any positive time T.

2. If $\sqrt{d} \notin \mathbb{Q}$, then System (6.4) is approximate controllable in $H^{-1}(0,\pi;\mathbb{R}^2)$ at any positive time T. With respect to the null controllability in $H^{-1}(0,\pi;\mathbb{R}^2)$ of this system, we have proved the existence of an optimal time $T_0 \in [0,\infty]$ (which depends on d) such that if $T < T_0$ System (6.4) is not null controllable at this time and if $T > T_0$ the system is null controllable at this time.

Remark 6.27. Let us now consider the boundary null controllability for the scalar heat equation,

$$\begin{cases} \partial_t y - \partial_{xx} y = 0 & \text{in } Q = (0,\pi) \times (0,T), \\ y(0,\cdot) = v, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases} \tag{6.22}$$

where $T > 0$ and $y_0 \in H^{-1}(0,\pi)$ are given and $v \in L^2(0,T)$ is a boundary control. This problem was considered in [10] and [11]. Following the ideas of Subsections 6.1 and (6.2), it is not difficult to see that Theorem 2.6 can be applied. In this case, from Theorem 4.8, we have $T_0 = c(\Lambda) = 0$, where $\Lambda = \{k^2\}_{k\geq 1}$ and the minimal time T_0 and the condensation index $c(\Lambda)$ are respectively given by (2.13) and (3.2). Therefore, we deduce that the null controllability result for System (6.22) is valid for every positive time T. □

6.3 Pointwise null controllability of a parabolic system

In the last application of Theorem 2.6 we are going to combine the difficulties appearing in the examples of Subsections 6.1 and 6.2. Consider the following parabolic system

$$\begin{cases} \partial_t y - (D\partial_{xx} + A)\, y = f(x)Bv(t) & \text{in } Q = (0,\pi) \times (0,T), \\ y(0,\cdot) = y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases} \tag{6.23}$$

where D, A and B are given in (6.5), $f \in H^{-1}(0,\pi)$ is a given function and $y_0 \in L^2(0,\pi;\mathbb{R}^2)$ is the initial datum. Again, $v \in L^2(0,T)$ is a scalar control which is exerted via the right-hand side of the system.

System (6.23) is well-posed. In fact, one has:

Proposition 6.28. *Let us consider the matrix D given in (6.5) and $A \in \mathcal{L}(\mathbb{R}^2)$ and $B \in \mathbb{R}^2$. Let $f \in H^{-1}(0,\pi)$ be a given function. Then, for any $y_0 \in L^2(0,\pi;\mathbb{R}^2)$ and $v \in L^2(0,T)$, System (6.23) possesses a unique weak solution y satisfying $y \in L^2(0,T;H_0^1(0,\pi;\mathbb{R}^2)) \cap C^0([0,T];L^2(0,\pi;\mathbb{R}^2))$ and*

$$\|y\|_{L^2(0,T;H_0^1(0,\pi;\mathbb{R}^2))} + \|y\|_{C^0([0,T];L^2(0,\pi;\mathbb{R}^2))} \leq C \left(\|y_0\|_{L^2(0,\pi;\mathbb{R}^2)} + \|f\|_{H^{-1}(0,\pi)} \|v\|_{L^2(0,T)} \right),$$

where C is a positive constant only depending on D, A and B. □

The proof of this result is similar to the proof of Proposition 6.1 and will be omitted.

As will be seen, the controllability properties of System (6.23) are different when $d = 1$ and $d \neq 1$. Let us first study these controllability properties in the more complicated case $d \neq 1$.

System (6.23) enters the framework of Section 2 by setting $\mathbb{X} = L^2(0,\pi;\mathbb{R}^2)$, $\mathcal{A}$ given by (6.7) and

$$\mathcal{B} = f(\cdot)B \in \mathcal{L}\left(\mathbb{R}, H^{-1}(0,\pi;\mathbb{R}^2)\right).$$

When $d \neq 1$ and taking into account Proposition 6.12, we can write

$$\mathcal{A} = -\sum_{k\geq 1} \left[\lambda_{1,k}(\cdot, \psi_{1,k})_{\mathbb{X}}\phi_{1,k} + \lambda_{2,k}(\cdot, \psi_{2,k})_{\mathbb{X}}\phi_{2,k}\right],$$

where the sequence $\{\lambda_{1,k}, \lambda_{2,k}\}_{k\geq 1}$ is given by (6.10) and

$$\{\phi_{1,k}, \phi_{2,k}\}_{k\geq 1} := \{V_{1,k}\Phi_k, V_{2,k}\Phi_k\}_{k\geq 1}, \quad \{\psi_{1,k}, \psi_{2,k}\}_{k\geq 1} := \{W_{1,k}\Phi_k, W_{2,k}\Phi_k\}_{k\geq 1},$$

(the vectors $V_{1,k}$, $V_{2,k}$, $W_{1,k}$ and $W_{2,k}$ are given in Propositions 6.11 and 6.12 and $\Phi_k(x)$ in (1.3)). Following the proof of Proposition 6.12, we also have that the set $\{\phi_{1,k}, \phi_{2,k}\}_{k\geq 1}$ is a Riesz basis of eigenfunctions of $\mathcal{A}$ in $\mathbb{X} = L^2(0,\pi;\mathbb{R}^2)$ and its biorthogonal basis is $\{\psi_{1,k}, \psi_{2,k}\}_{k\geq 1}$. Observe also that the sequence of eigenvalues of $-\mathcal{A}$, $\{-\lambda_{1,k}, -\lambda_{2,k}\}_{k\geq 1}$, satisfies assumption (2.2).

On the other hand, we also have

$$\mathbb{X}_1 = D(\mathcal{A}) = H^2(0,\pi;\mathbb{R}^2) \cap H_0^1(0,\pi;\mathbb{R}^2) \quad \text{and} \quad \mathbb{X}_{-1} = \left(H^2(0,\pi;\mathbb{R}^2) \cap H_0^1(0,\pi;\mathbb{R}^2)\right)',$$

and, therefore, $\mathcal{B} \in \mathcal{L}(\mathbb{R}, \mathbb{X}_{-1})$. Finally, from Proposition 6.28 we readily obtain that the operator $\mathcal{B}$ is an admissible control operator for the semigroup generated by $\mathcal{A}$. Furthermore, with the notations of Subsection 6.2, we have:

$$\langle \mathcal{B}v, \psi_{i,k}\rangle_{\mathbb{X}_{-1},\mathbb{X}_1} = vB^*W_{i,k}\langle f, \Phi_k\rangle,$$

and thus

$$\mathcal{B}^*\psi_{i,k} = B^*W_{i,k}\langle f, \Phi_k\rangle, \quad \forall k \geq 1,\ i = 1, 2,$$

where $\langle\cdot,\cdot\rangle$ is the duality pairing between $H^{-1}(0,\pi)$ and $H_0^1(0,\pi)$.

The previous considerations together with Corollary 2.4 permit to state the approximate controllability result for System (6.23):

Theorem 6.29. *Assume $d \neq 1$. Under the previous assumptions, System (6.23) is approximately controllable in $\mathbb{X} = L^2(0,T;\mathbb{R}^2)$ at time T if and only if*

$$\begin{cases} \sqrt{d} \notin \mathbb{Q}, \quad b_{1,k} := B^*W_{1,k}\langle f, \Phi_k\rangle = \left[b_1 + \dfrac{b_2}{(d-1)k^2}\right]\langle f, \Phi_k\rangle \neq 0, \\ b_{2,k} := B^*W_{2,k}\langle f, \Phi_k\rangle = b_2\langle f, \Phi_k\rangle \neq 0, \quad \forall k \geq 1. \end{cases} \tag{6.24}$$

Proof. The proof of this result can be obtained following the ideas of the proof of Theorem 6.14. The details are left to the reader. □

Remark 6.30. The conditions in (6.24) can be written in the shorter way:

$$\sqrt{d} \notin \mathbb{Q} \quad \text{and} \quad b_2\left[b_1(d-1)k^2 + b_2\right]\langle f, \Phi_k\rangle \neq 0, \quad \forall k \geq 1.$$

Again, these conditions can be seen as a Kalman condition for the approximate controllability of System (6.23) at time T. In this case, conditions (6.24) are equivalent to the properties

1. The eigenvectors of the operator $\mathcal{A}$ are simple (geometric multiplicity 1).
2. The ordinary differential system

$$\begin{cases} y_k' = (-k^2D + A)y_k + \langle f, \Phi_k\rangle Bv \quad \text{on } (0,T), \\ y_k(0) = y_{0,k} \in \mathbb{R}^2. \end{cases}$$

is controllable for any k. □

As a consequence of Theorem 2.6, we have the following result:

Theorem 6.31. *Let us consider the matrices D, A and B given by (6.5) with $d \neq 1$. In addition, assume that conditions (6.24) hold. Let us take*

$$T_0 = \max_{i=1,2} \limsup \left(\frac{\log 1/|b_{i,k}| + \log 1/|E'(\lambda_{i,k})|}{\lambda_{i,k}} \right),$$

where $\Lambda = \{\Lambda_\ell\}_{\ell \geq 1} := \{\lambda_{1,k}, \lambda_{2,k}\}_{\ell \geq 1}$ and $b_{i,k}$ are given in (6.10) and (6.24), and

$$E(z) = \prod_{\ell=1}^{\infty} \left(1 - \frac{z^2}{\Lambda_\ell^2} \right), \quad z \in \mathbb{C}.$$

Then:

1. *System (6.23) is null controllable in $\mathbb{X} = L^2(0, \pi; \mathbb{R}^2)$ for $T > T_0$;*
2. *System (6.23) is not null controllable in $\mathbb{X} = L^2(0, \pi; \mathbb{R}^2)$ for $T < T_0$.*

Proof. The proof can be deduced from Theorem 2.6, Proposition 6.11 and Proposition 6.12. The details are left to the reader. □

In order to finish this subsection, let us consider System (6.23) with D, A and B be given by (6.5) in the simplest case $d = 1$. In this case, the eigenvalues of the operator $-\mathcal{A}$ (see (6.7)) are given by $\{k^2\}_{k \geq 1}$ and the corresponding eigenfunctions are given by

$$\phi_k = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Phi_k, \quad \forall k \geq 1.$$

Observe that the set of eigenfunctions of $-\mathcal{A}$ is not a Riesz basis of $\mathbb{X} = L^2(0, \pi; \mathbb{R}^2)$ and therefore Theorem 2.6 cannot be applied. Nevertheless, the controllability properties of System (6.23) can be deduced from the results stated in [10, 9, 12, 3]. If we denote

$$f_k := \langle f, \Phi_k \rangle, \quad \forall k \geq 1, \tag{6.25}$$

one has:

Theorem 6.32. *Let us consider the matrices D, A and B given by (6.5) with $d = 1$. Then,*

1. *System (6.23) is approximately controllable in $\mathbb{X} = L^2(0, \pi; \mathbb{R}^2)$ at time T if and only if*

$$b_2 f_k = b_2 \langle f, \Phi_k \rangle \neq 0, \quad \forall k \geq 1. \tag{6.26}$$

2. *Assume that (6.26) holds and take*

$$T_1 := \limsup \frac{\log 1/|f_k|}{k^2} \in [0, \infty].$$

Then,

   (a) *System (6.23) is null controllable in $\mathbb{X} = L^2(0, \pi; \mathbb{R}^2)$ for $T > T_1$;*
   (b) *System (6.23) is not null controllable in $\mathbb{X} = L^2(0, \pi; \mathbb{R}^2)$ for $T < T_1$.*

Proof. Observe that for $d=1$, we have $D=Id$. The proof is then a consequence of previous results proved in [12] and [3].

As saw before, the controllability properties of System (6.23) are equivalent to appropriate properties of the adjoint system

$$\begin{cases} -\partial_t\varphi = (Id\partial_{xx} + A^*)\,\varphi, & \text{in } Q,\\ \varphi(0,\cdot)=\varphi(\pi,\cdot)=0 & \text{on } (0,T),\\ \varphi(\cdot,T)=\varphi_0 & \text{in } (0,\pi), \end{cases} \tag{6.27}$$

where $\varphi_0 \in L^2(0,\pi;\mathbb{R}^2)$ is given. In this case, it is not difficult to construct the solutions to the adjoint system (6.27). Indeed, if

$$\varphi_0(x) = \sum_{k\geq 1} \begin{pmatrix} a_k \\ b_k \end{pmatrix} \Phi_k(x), \quad \text{a.e. } x\in(0,\pi),$$

where $\{a_k\}_{k\geq 1}, \{b_k\}_{k\geq 1} \in \ell^2$, then the solution φ to the adjoint system (6.27) is given by

$$\varphi(x,t) = \sum_{k\geq 1} e^{-k^2(T-t)} \begin{pmatrix} 1 & 0 \\ T-t & 1 \end{pmatrix} \begin{pmatrix} a_k \\ b_k \end{pmatrix} \Phi_k(x), \quad \text{a.e. } (x,t)\in Q.$$

Therefore

$$B^*\langle f, \varphi(\cdot,t)\rangle = \sum_{k\geq 1} e^{-k^2(T-t)} \left[A_k(T-t)+B_k\right] f_k, \quad \text{a.e. } t\in(0,T), \tag{6.28}$$

where f_k is given by (6.25) and

$$\begin{pmatrix} A_k \\ B_k \end{pmatrix} := \begin{pmatrix} b_2 & 0 \\ b_1 & b_2 \end{pmatrix} \begin{pmatrix} a_k \\ b_k \end{pmatrix}, \quad \forall k\geq 1. \tag{6.29}$$

We will also need the following result:

Lemma 6.33. *For every $T>0$ and $\varepsilon>0$, there exists a positive constant $C(\varepsilon,T)$ such that*

$$\sum_{k\geq 1} \frac{1}{k^2}\left(|A_k|^2+|B_k|^2\right) e^{-2\varepsilon k^2} \leq C(\varepsilon,T) \int_0^T \left| \sum_{k\geq 1} (A_k t + B_k) e^{-k^2 t} \right|^2 dt$$

for any sequences $\{A_k\}_{k\geq 1}, \{B_k\}_{k\geq 1} \in \ell^2$. □

The proof of this result is implicitly given in [12], (see the proof of Proposition 3.4, p. 1727–1728). We are now ready to prove the theorem.

1. The approximate controllability result for system (6.23) in $L^2(0,\pi;\mathbb{R}^2)$ at time $T>0$ is equivalent to the following unique continuation property for the solutions $\varphi \in C^0([0,T];L^2(0,\pi;\mathbb{R}^2))$:

"*If $B^*\langle f,\varphi(\cdot,t)\rangle = 0$ for almost all $t\in(0,T)$, then $\varphi_0=0$.*"

From (6.28) and (6.29) we deduce that (6.26) is necessary.

On the other hand, if (6.26) holds, the solutions to the adjoint problem (6.27) satisfies the previous unique continuation property. Indeed, if $B^*\langle f,\varphi(\cdot,t)\rangle = 0$ for almost all $t\in(0,T)$, from (6.28) and Lemma 6.33 (applied to f_kA_k, f_kB_k and $\varepsilon = T$), we deduce $f_kA_k = f_kB_k = 0$ for any $k\geq 1$.

Using (6.29) and assumption (6.26) we get $a_k = b_k = 0$ for any $k \geq 1$, i.e., $\varphi_0 \equiv 0$. This proves the first point of Theorem 6.32.

2. Let us now prove the second point of Theorem 6.32 and let us assume that (6.26) holds. In this case, the null controllability result for System (6.23) is equivalent to the existence of a positive constant C_T such that the solutions to (6.27) satisfy the observability inequality:

$$\|\varphi(\cdot,0)\|^2_{L^2(0,\pi;\mathbb{R}^2)} \leq C_T \int_0^T \left| B^* \langle f, \varphi(\cdot,t)\rangle \right|^2 \, dt$$

where $\varphi_0 \in L^2(0,\pi;\mathbb{R}^2)$. Using the expression of φ and (6.28)–(6.29), the previous observability inequality amounts to the existence of a positive constant C_T such that

$$\sum_{k\geq 1} e^{-2k^2T} \left| \begin{pmatrix} 1 & 0 \\ T & 1 \end{pmatrix} \begin{pmatrix} a_k \\ b_k \end{pmatrix} \right|^2_{\mathbb{R}^2} \leq C_T \int_0^T \left| \sum_{k\geq 1} e^{-k^2 t}(A_k t + B_k) f_k \right|^2 \, dt, \tag{6.30}$$

for any sequences $\{a_k\}_{k\geq 1}, \{b_k\}_{k\geq 1} \in \ell^2$, and where A_k and B_k are given by (6.29).

(a) Let us assume that $T > T_1$, with $T_1 \in [0,\infty)$ given in the statement of the theorem. Let us fix $\varepsilon > 0$. From the definition of the time T_1 we deduce

$$|f_k| \geq C_\varepsilon e^{-k^2(T_1+\varepsilon)}, \quad \forall k \geq 1,$$

where $C_\varepsilon > 0$ is a constant. On the other hand, from Lemma 6.33 applied to the sequences $\{f_k A_k\}_{k\geq 1}$ and $\{f_k B_k\}_{k\geq 1}$, we infer

$$\begin{aligned}
\int_0^T \left| \sum_{k\geq 1} e^{-k^2 t}(A_k t + B_k) f_k \right|^2 \, dt &\geq C(\varepsilon,T) \sum_{k\geq 1} \frac{1}{k^2} \left(|f_k A_k|^2 + |f_k B_k|^2 \right) e^{-2\varepsilon k^2} \\
&\geq C(\varepsilon,T) \sum_{k\geq 1} \frac{1}{k^2} \left(|A_k|^2 + |B_k|^2 \right) e^{-2k^2(T_1+2\varepsilon)} \\
&\geq C(\varepsilon,T) \sum_{k\geq 1} \left(|A_k|^2 + |B_k|^2 \right) e^{-2k^2(T_1+3\varepsilon)}.
\end{aligned}$$

Finally, let us take $\varepsilon = (T - T_1)/3$. Thus, from (6.29) ($b_2 \neq 0$), the previous inequality gives

$$\int_0^T \left| \sum_{k\geq 1} e^{-k^2 t}(A_k t + B_k) f_k \right|^2 \, dt \geq C(T) \sum_{k\geq 1} e^{-2k^2 T} \left(|a_k|^2 + |b_k|^2 \right),$$

and, evidently, (6.30). This proves the point (a).

(b) Let us now assume that $T < T_1 \in (0,\infty]$. This point is a direct consequence of the results stated in [9]. Indeed, taking $a_k = 0$ for any $k \geq 1$, inequality (6.30) transforms into

$$\sum_{k\geq 1} e^{-2k^2 T} |b_k|^2 \leq C_T |b_2| \int_0^T \left| \sum_{k\geq 1} e^{-k^2 t} b_k f_k \right|^2 \, dt, \tag{6.31}$$

where $\{b_k\}_{k\geq 1} \in \ell^2$.

Let us see that inequality (6.31) fails when $T < T_1$. To this end, let us take $\varepsilon > 0$ such that $T + \varepsilon < T_1$. From the definition of T_1 we deduce that the series

$$\sum_{k\geq 1} \frac{e^{-k^2(T+\varepsilon)}}{|f_k|}$$

diverges. Thus, for a subsequence $\{k_n\}_{n\geq 1}$ we must have

$$\frac{e^{-k_n^2(T+\varepsilon)}}{|f_{k_n}|} \geq \frac{1}{k_n^2}, \quad \forall n \geq 1.$$

Multiplying this last inequality by k_n^2 we deduce that for a positive constant C_ε, one has

$$\frac{e^{-k_n^2 T}}{|f_{k_n}|} \geq k_n^2 \frac{e^{-k_n^2(T+\varepsilon)}}{|f_{k_n}|} \geq C_\varepsilon, \quad \forall n \geq 1.$$

Finally, let us take the sequence $\{b_k^{(n)}\}_{k\geq 1}$ given by

$$b_k^{(n)} = \begin{cases} \dfrac{1}{f_{k_n}} & \text{if } k = k_n \\ 0 & \text{otherwise.} \end{cases}$$

From inequality (6.31) applied to the previous sequence $\{b_k^{(n)}\}_{k\geq 1}$ we deduce that for any $n \geq 1$ one has

$$C_\varepsilon^2 \leq \frac{e^{-2k_n^2 T}}{|f_{k_n}^2|} \leq C_T |b_2| \int_0^T e^{-2k_n^2 t}\, dt = C_T |b_2| \frac{1}{2k_n^2}\left(1 - e^{-2k_n^2 T}\right).$$

With this last inequality we evidently obtain a contradiction. Therefore, inequalities (6.31) and, of course, (6.30) fail. This proves the statement (b) and finalizes the proof of the result. □

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A Proofs of Lemma 6.22 and Corollary 6.25

This appendix will be devoted to proving Lemma 6.22 and Corollary 6.25. To this end, we will use some results from the diophantine approximation theory. Thus, in a first section we will recall some known properties of continued fractions. We will devote the second section to proving the two results.

A.1 Some basic properties of continued fractions

We refer for this subsection to [22].

Let us recall that a (simple) continued fraction is an expression of the form

$$[a_0, a_1, \dots, a_n] = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\dots + \frac{1}{a_n}}}},$$

where $a_0, a_1, \dots, a_n$ are real numbers satisfying $a_i \neq 0$ for any $i \geq 1$. By induction:

$$\begin{cases} [a_0] = a_0 \\ [a_0, a_1] = a_0 + \frac{1}{a_1} \\ [a_0, a_1, \dots, a_n] = \left[a_0, a_1, \dots, a_{n-2}, a_{n-1} + \frac{1}{a_n}\right], \quad n \geq 1. \end{cases}$$

Let us take $a_0 \in \mathbb{Z}$ and $\{a_k\}_{k\geq 1} \subset \mathbb{N}$ a sequence of positive integers. Then, it is easy to see that $[a_0, a_1, \dots, a_n] \in \mathbb{Q}$ for all $n \geq 0$. On the other hand, if $x \in \mathbb{Q}$ there exist $n \in \mathbb{N}$ and integer numbers $a_0, a_1, \dots, a_n$, with $a_k \geq 1$ for any $1 \leq k \leq n$, such that

$$[a_0, a_1, \dots, a_n] = x.$$

Let us point out the previous representation of the rational number x is not unique. Indeed, it is easy to check

$$x = \begin{cases} [a_0, a_1, \dots, a_n] = [a_0, a_1, \dots, a_n - 1, 1] & \text{if } a_n \geq 2, \\ [a_0, a_1, \dots, a_{n-1}, 1] = [a_0, a_1, \dots, a_{n-1} + 1] & \text{if } a_n = 1. \end{cases}$$

Let us now recall some classical properties of simple continued fractions. One has:

Theorem A.1. *Let us fix $a_0 \in \mathbb{Z}$ and $\{a_k\}_{k\geq 1} \subset \mathbb{N}$ a sequence of positive integers.*

1. *If $p_{-1} = 1$, $p_0 = a_0$, $q_{-1} = 0$ and $q_0 = 1$ and*

$$p_n = a_n p_{n-1} + p_{n-2}, \quad q_n = a_n q_{n-1} + q_{n-2}, \quad \forall n \geq 1, \tag{A.1}$$

   then,

$$[a_0, a_1, \dots, a_n] = \frac{p_n}{q_n}, \quad n \geq 0.$$

2. *The sequence $\{q_n\}_{n\geq 1}$ is strictly increasing and $q_n > 0$ for all $n \geq 0$. Moreover, if $a_0 \geq 0$, then $p_n \geq 1$, for any $n \geq 1$, and the sequence $\{p_n\}_{n\geq 0}$ is also strictly increasing.*

3. *The following identities hold:*

$$\begin{cases} \dfrac{p_n}{q_n} - \dfrac{p_{n-1}}{q_{n-1}} = \dfrac{(-1)^{n+1}}{q_n q_{n-1}}, & \forall n \geq 1, \\ \dfrac{p_n}{q_n} - \dfrac{p_{n-2}}{q_{n-2}} = \dfrac{(-1)^{n}}{q_n q_{n-2}} a_n, & \forall n \geq 2. \end{cases} \tag{A.2}$$

   In particular, $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1}$ and p_n and q_n are co-prime for any $n \geq 0$. $\square$

The second group of properties for simple continued fractions concerns the approximation of real numbers. One has:

Theorem A.2. *Let us fix $a_0 \in \mathbb{Z}$ and $\{a_k\}_{k\geq 1} \subset \mathbb{N}$ a sequence of positive integers. Then,*

1. *The sequence of simple continued fractions $\{[a_0, a_1, \dots, a_n]\}_{n\geq 1}$ is convergent in $\mathbb{R}$ to an irrational number x. Conversely, for all $x \in \mathbb{R} \setminus \mathbb{Q}$, there exist a unique integer $a_0 \in \mathbb{Z}$ and a unique sequence $\{a_i\}_{i\geq 1} \subset \mathbb{N}$ of positive integers such that $\lim \frac{p_n}{q_n} = x$, where p_n and q_n are given by (A.1), i.e.,*

$$\frac{p_n}{q_n} = [a_0, a_1, \dots, a_n], \quad \forall n \geq 0.$$

   In this case, we will write $x = [a_0, a_1, \dots, a_n, \dots]$. The rational number p_n/q_n is called the n-th convergent of x.

2. *If* $x = [a_0, a_1, \ldots, a_n, \ldots]$ *and* p_n/q_n *is the corresponding* n*-th convergent of* x*, then*

$$\frac{1}{q_n\left(q_{n+1}+q_n\right)} < \left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}}, \quad \forall n \geq 0, \tag{A.3}$$

and the sequence $\left\{\frac{p_{2n}}{q_{2n}}\right\}_{n\geq 0}$ *(resp.,* $\left\{\frac{p_{2n+1}}{q_{2n+1}}\right\}_{n\geq 0}$*) is increasing (resp., decreasing). Moreover, the convergents of* x *satisfies* $\lim |xq_n - p_n| = 0$ *and*

$$|xq - p| \geq |xq_n - p_n|, \quad \forall n \geq 1, \quad \forall p, q \in \mathbb{Z}, \quad \text{with } 1 \leq q \leq q_n, \tag{A.4}$$

with equality if and only if $p = p_n$ *and* $q = q_n$ *(it is said that the convergents of* x *are the best approximation of* x *of the second kind).* □

Let us note that from (A.2), it follows that

$$[a_0, a_1, \ldots, a_n] = a_0 + \sum_{k=1}^{n} \left(\frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}} \right) = a_0 + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{q_k q_{k-1}}, \quad \forall n \geq 0.$$

Thus,

$$[a_0, a_1, \ldots, a_n, \ldots] = a_0 + \sum_{k\geq 1} \frac{(-1)^{k+1}}{q_k q_{k-1}}.$$

A.2 Proofs of the results

Let us first prove the first part of Lemma 6.22 and Corollary 6.25 when $\tau_0 \in (0,\infty)$. To this end, let us fix $\tau_0 \in (0,\infty)$, $x_0 \in (0,\infty)$ and $\varepsilon > 0$. The objective will be to determine an integer $a_0 \geq 0$ and a sequence $\{a_k\}_{k\geq 1}$ of positive integer numbers such that the infinite simple continued fraction $\sqrt{d} := [a_0, a_1, \ldots, a_n, \ldots]$ satisfies (6.18) and (6.19), for p_k and q_k given by (A.1), and

$$|\sqrt{d} - x_0| \leq \varepsilon.$$

Let us fix $a_0 \in \mathbb{Z}$ and a sequence $\{a_k\}_{k\geq 1} \subset \mathbb{N}$ of positive integers such that $a_0 \geq 0$ and

$$\lim a_k = \infty. \tag{A.5}$$

Consider the sequence of convergents $\left\{\frac{p_n}{q_n}\right\}_{n\geq 0}$, defined by (A.1), of

$$\nu = [a_0, a_1, \ldots, a_n, \ldots] = \lim \frac{p_n}{q_n} = \lim\, [a_0, a_1, \ldots, a_n]\,.$$

Multiplying inequality (A.3) by $a_{n+1}q_n^2$, $n \geq 0$, we have:

$$\frac{a_{n+1}q_n}{q_{n+1}+q_n} < a_{n+1}q_n^2 \left| \nu - \frac{p_n}{q_n} \right| < \frac{a_{n+1}q_n}{q_{n+1}}, \quad \forall n \geq 0.$$

On the other hand, from (A.1), we also deduce:

$$1 - \frac{q_n + q_{n-1}}{q_{n+1}+q_n} < a_{n+1}q_n^2 \left| \nu - \frac{p_n}{q_n} \right| < 1 - \frac{q_{n-1}}{q_{n+1}}, \quad \forall n \geq 0. \tag{A.6}$$

Using once again (A.1), we obtain

$$\frac{q_{n+1}}{q_n} = a_{n+1} + \frac{q_{n-1}}{q_n}, \quad \forall n \geq 1.$$

So that, since $\lim a_n = \infty$ and the sequence $\{q_n\}_{n\geq 0}$ is increasing , it follows that

$$\lim \frac{q_{n+1}}{q_n} = \infty \quad \text{and} \quad \lim \frac{q_{n+1}}{q_{n-1}} = \infty.$$

Thus,

$$\lim \frac{q_n + q_{n-1}}{q_{n+1} + q_n} = 0,$$

and coming back to (A.6) we arrive to

$$\lim \left(a_{n+1} q_n^2 \left| \nu - \frac{p_n}{q_n} \right| \right) = 1.$$

Taking into account the previous identity, given $\tau_0 \in (0, \infty)$, the task consists in finding an integer $a_0 \geq 0$ and a sequence $\{a_k\}_{\geq 1} \subset \mathbb{N}$ satisfying (A.5) for which

$$\lim \frac{a_{n+1} q_n^2}{e^{\tau_0 p_n^2}} = 1. \tag{A.7}$$

Taking $\sqrt{d} = \nu = [a_0, a_1, \ldots, a_n, \ldots]$, we obtain the proof of (6.18).

We reason as follows: Let us fix $k_0 \in \mathbb{N}$, $a_0 \in \mathbb{Z}$ and positive integers $\{a_1, \ldots, a_{k_0}\}$ such that $a_0 \geq 0$ and

$$\frac{1}{k_0(k_0+1)} \leq \frac{\varepsilon}{2}, \quad |[a_0, a_1, \ldots, a_{k_0}] - x_0| \leq \frac{\varepsilon}{2}.$$

On the other hand, let us take $p_{-1} = 1$, $p_0 = a_0$, $q_{-1} = 0$ and $q_0 = 1$. Following formula (A.1) we can construct the convergents p_k/q_k for any $k : 1 \leq k \leq k_0$. Now, let us set

$$a_{k_0+1} = \max\left\{1, \left\lfloor e^{\tau_0 p_{k_0}^2} / q_{k_0}^2 \right\rfloor \right\},$$

where $\lfloor \cdot \rfloor$ is the floor function (for $x \in \mathbb{R}$, $\lfloor x \rfloor$ gives the largest integer less than or equal to x). Again, formula (A.1) allows us to compute p_k and q_k for $k : 1 \leq k \leq k_0 + 1$.

We can continue the argument reasoning by induction: Given $n \geq k_0 + 1$ and nonnegative integers a_k, with $0 \leq k \leq n$, we calculate p_k and q_k using (A.1), $(0 \leq k \leq n)$ and take

$$a_{n+1} = \max\left\{1, \left\lfloor e^{\tau_0 p_n^2} / q_n^2 \right\rfloor \right\}.$$

Clearly, the sequence $\{a_k\}_{k\geq 0} \subset \mathbb{N}$ satisfies $a_k \geq 1$, for any $k \geq 1$, and then, $\lim p_n = \lim q_n = \infty$ and $\lim (p_n/q_n) = \nu$, where $\nu = [a_0, a_1, \ldots, a_n, \ldots]$. So,

$$\lim \frac{e^{\tau_0 p_n^2}}{q_n^2} = \lim \frac{e^{\tau_0 p_n^2}}{p_n^2} \frac{p_n^2}{q_n^2} = \infty,$$

whence

$$\lim a_{n+1} = \lim \left\lfloor \frac{e^{\tau_0 p_n^2}}{q_n^2} \right\rfloor = \infty.$$

In conclusion, the sequence $\{a_k\}_{k\geq 0} \subset \mathbb{N}$ satisfies (A.5) and, therefore, (A.7). This proves (6.18).

We can also write

$$\begin{aligned} |\nu - x_0| &\leq |\nu - [a_0, a_1, \ldots, a_{k_0}]\,| + |[a_0, a_1, \ldots, a_{k_0}] - x_0| \\ &\leq \frac{1}{q_{k_0} q_{k_0+1}} + \frac{\varepsilon}{2} \leq \frac{1}{k_0(k_0+1)} + \frac{\varepsilon}{2} \leq \varepsilon. \end{aligned}$$

In the previous inequality we have used (A.3) and the property $q_k \geq k$, for any $k \geq 1$, which is valid for any sequence $\{a_k\}_{k\geq 0} \subset \mathbb{Z}$ with $a_0 \geq 0$ and $a_k > 0$ for $k \geq 1$. This also proves Corollary 6.25 when $\tau_0 \in (0, \infty)$.

The inequality (6.19) can be directly deduced from the property (A.4). This ends the first part of Lemma 6.22.

The second part of Lemma 6.22 follows in the same way by constructing a sequence $\{a_k\}_{k\geq 1} \subset \mathbb{N}$ such that, for a given $\sigma > 0$, one has

$$\lim \frac{e^{p_n^{2+\sigma}}}{a_{n+1} q_n^2} = 0,$$

and this can be done choosing (for instance)

$$a_{n+1} = \max \left\{ 1, \left\lfloor \frac{e^{p_n^{2+\sigma}}}{q_n} \right\rfloor, \right\} \quad \forall n \geq k_0.$$

This finalizes the proof of Lemma 6.22.

Finally, when $\tau_0 = \infty$, Corollary 6.25 can be proved following the previous ideas. $\square$